Generalized Flexural Design Equations for Prestressed Concrete

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Systematic design equations for prestressed concrete flexural members have been presented previously by Guyon,[1] Nilson,[2,3] and Naaman.[4,5] While of significant value, the equation sets presented cannot be applied directly to all steel profiles and loading systems. The set presented by Naaman will define the minimum required beam section moduli, at a given section (normally the point of maximum moment), without consideration of the steel profile. Often, use of these minimum moduli will require a draped strand configuration.

Nilson presents essentially the same set of equations, recognizing that a steel profile with varying eccentricity along the span will normally be required. A second set is also presented, which was derived specifically for a constant eccentricity steel profile. The equation sets in Refs. 2 through 5 are thus not directly applicable to design with other common steel profiles such as strands deflected at midspan.

The design equations presented here were developed employing the same techniques and principles used previously. However, their development accounts for variables not considered before, and the resulting equation set is directly applicable to the design of general noncomposite prestressed concrete flexural members.

The steel profile is considered directly in the application of the equations, and the equations may be used for design with full or partial prestressing. The equation sets may be used for selecting minimum weight standard sections, quickly investigating the effect of different steel profiles on section moduli requirements, and developing new or nonstandard sections for a given application.

Synopsis

General design equations are developed for the systematic design of prestressed concrete flexural members.

The set of equations presented may be applied directly to the design of prismatic prestressed beams regardless of the prestressing steel profile and loading. They are intended for use with allowable concrete stress limits and the working stress design method.

The equation set permits a systematic, direct design approach for general noncomposite prestressed concrete flexural members. Numerical examples are included to illustrate the application of the proposed equations.

EQUATION SET DEVELOPMENT

The use of these equations requires a preliminary estimate of the effectiveness ratio, R. This is defined[3] as the ratio of the effective prestressing force after all losses have occurred, P_e, to the initial prestress force immediately after transfer, P_i.

It is not, however, necessary to assume all losses will have occurred prior to the application of the superimposed loads. An estimate of the self weight, w_o, of the beam is also required, as is, occasionally, an estimate of the ratio of the distance from the section centroid to the top of the beam, c_1, to the total section depth, h.

Prestressed concrete beams designed by the working stress method presently in use must not develop flexural stresses that exceed the allowable stresses both at the time of force transfer and at the full service load stage. The "critical section" at each of these load stages is defined as the cross section at which the concrete stresses are highest relative to the allowable stresses.

For beams with a constant steel eccentricity, the critical sections at transfer are the beam ends, where the self weight moment, M_o, does not counter the prestressing moment. For beams having a strand profile that is deflected at discrete points within the span, the critical section at transfer normally occurs at one or more of the deflection or drape points (the end eccentricity being limited to ensure this).

The critical section under service loading is located at the point having the highest ratio of total moment due to loads to the moment due to prestressing (if, it is assumed, a prestressing force remains essentially constant throughout the span). Normally, this is located at or very near the point of maximum total moment due to loads (including self weight).

INITIAL PRESTRESSING FORCE, P_i

Fig. 1 illustrates the limiting stress distribution at the critical section at transfer. The allowable tensile and compressive stresses, f_{ti} and f_{ci}, respectively, are both attained. Note that the sign convention adopted herein assumes that compressive stresses are negative.

The compressive stress at the section centroid, f_{cg}, may be computed from the following equation developed from consideration of similar triangles:[3]

$$f_{cg} = f_{ti} - \frac{c_1}{h} (f_{ti} - f_{ci}) \qquad (1)$$

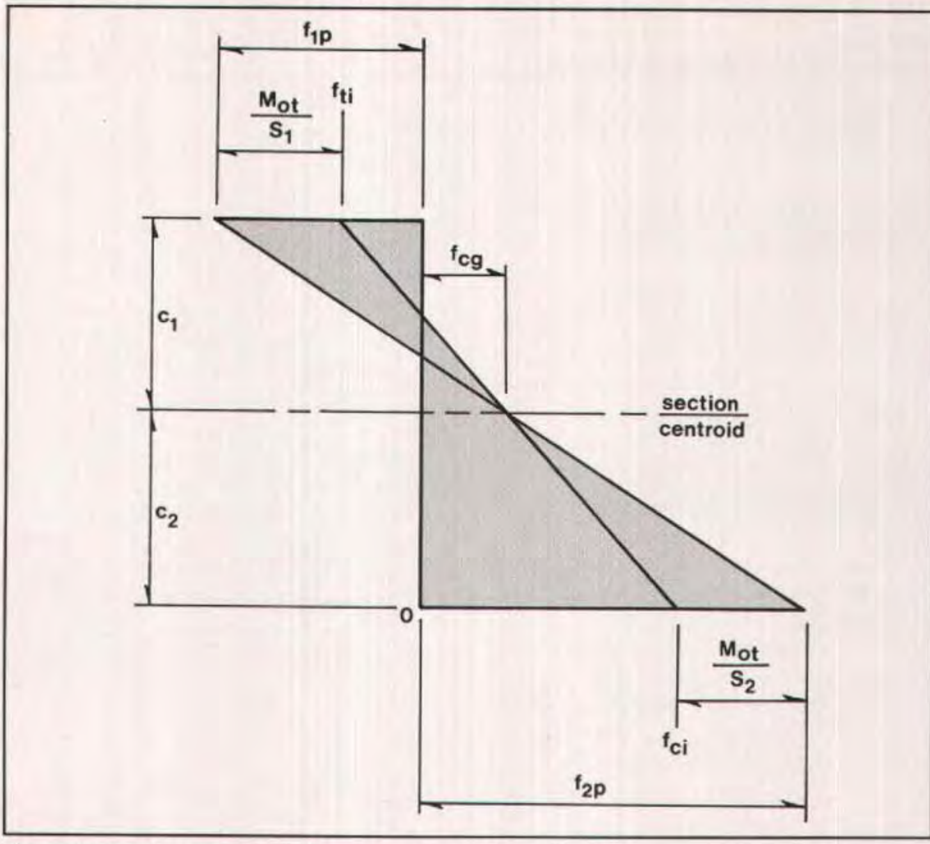

Fig. 1. Optimal stress distribution at critical section at transfer.

The initial prestressing force at transfer is then given by:

$$P_i = -A_c f_{cg} \tag{2}$$

where A_c is the gross cross-sectional area of the beam.

MAXIMUM ECCENTRICITY AT CRITICAL SECTION AT TRANSFER

The self weight moment at the critical section at transfer is defined as M_{ot}, and the limiting stress distribution at the critical section shown in Fig. 1 applies. In Fig. 1, f_{1p} and f_{2p} are the top and bottom fiber stresses, respectively, that would occur due to prestressing alone. The symbols S_1 and S_2 are the upper and lower elastic section moduli, respectively.

At the top of the section, the allowable tensile stress f_{ti} is given by:

$$f_{ti} = -\frac{P_i}{A_c} + \frac{P_i e_t}{S_1} - \frac{M_{ot}}{S_1} \tag{3}$$

where e_t is the steel eccentricity at the critical section at transfer. Upon substitution of Eq. (2) and rearranging, the maximum eccentricity is given as:

$$(e_t)_{max} = \frac{S_1}{P_i}\,(f_{ti} - f_{cg}) + \frac{M_{ot}}{P_i} \tag{4}$$

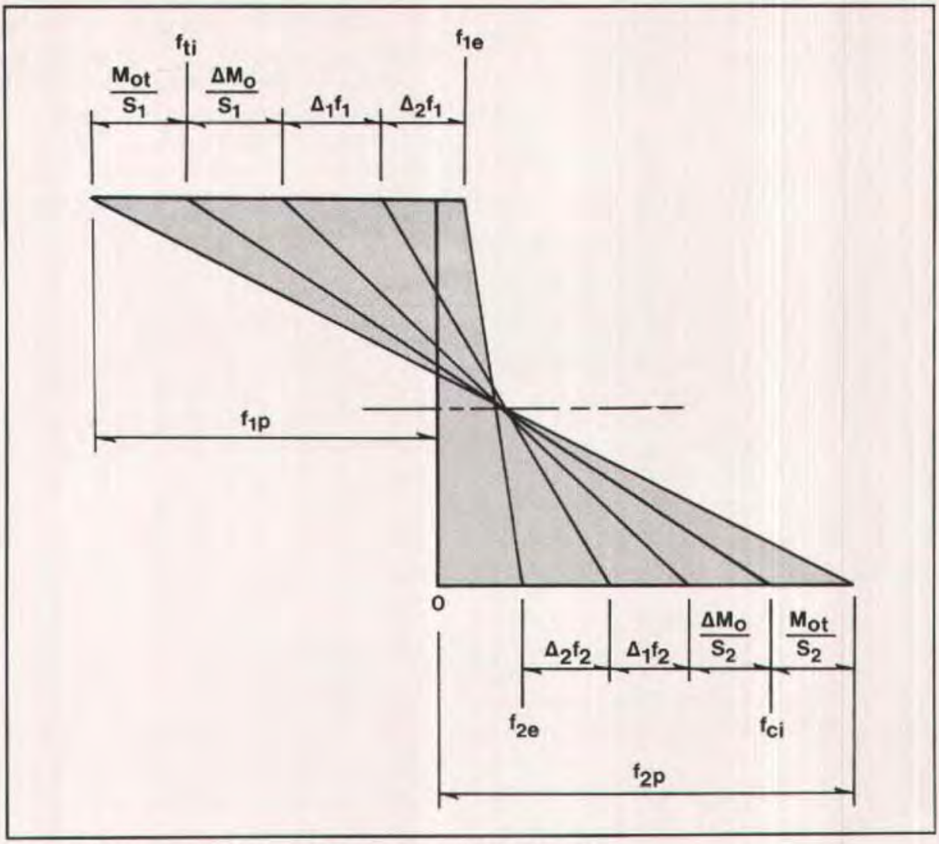

Fig. 2. Stress distributions at full service critical section.

REQUIRED SECTION MODULI

The stress distribution shown in Fig. 1 applies at the full service critical section prior to the application of the superimposed loads providing the self weight moment and eccentricity at the two critical sections are identical, and the effectiveness ratio, R, is unity.

In general, however, the stress distribution in Fig. 2 applies in which three possible stress increments have been included:

1. The stress increment due to changes in the self weight moment between the two critical sections.

2. The stress increment due to differing eccentricities between the critical sections.

3. The stress increment due to prestressing losses as the prestress force is reduced from P_i to P_e.

In Fig. 2, ΔM_o is the difference (positive or negative) in self weight moment between the critical sections, and is given by:

$$\Delta M_o = M_{os} - M_{ot} \qquad (5)$$

where M_{os} is the self weight moment at the full service critical section. The stress reductions (or increases) $\Delta_1 f_1$ and $\Delta_1 f_2$ are the top and bottom stress increments, respectively, due to differences in eccentricity between the two critical sections:

$$\Delta_1 f_1 = \frac{P_e e_t}{S_1} - \frac{P_e e_s}{S_1} = \frac{P_e}{S_1}(e_t - e_s) \quad (6)$$

and

$$\Delta_1 f_2 = \frac{P_e}{S_2}(e_t - e_s) \quad (7)$$

where e_t is the eccentricity at the critical transfer section and e_s is the eccentricity at the full service critical section. The quantity $(e_t - e_s)$ may again be either positive or negative. In many practical cases, this quantity is zero.

The stress reductions $\Delta_2 f_1$ and $\Delta_2 f_2$ are due to the reduction in prestress force from time-dependent losses. The top fiber stress range that would occur due to prestressing alone, f_{1p}, is given by:

$$f_{1p} = f_{ti} + \frac{M_{ot}}{S_1} \quad (8)$$

After time-dependent losses have occurred, this stress is reduced to:

$$\frac{P_e}{P_i} f_{1p} = R\left(f_{ti} + \frac{M_{ot}}{S_1}\right) \quad (9)$$

Hence:

$$\Delta_2 f_1 = f_{ti} + \frac{M_{ot}}{S_1} - R\left(f_{ti} + \frac{M_{ot}}{S_1}\right) \quad (10)$$

or

$$\Delta_2 f_1 = (1 - R)\left(f_{ti} + \frac{M_{ot}}{S_1}\right) \quad (11)$$

Similarly:

$$\Delta_2 f_2 = (1 - R)\left(-f_{ci} + \frac{M_{ot}}{S_2}\right) \quad (12)$$

The top and bottom stresses, f_{1e} and f_{2e}, respectively, are therefore the effective stresses at the full service load critical section prior to the application of the superimposed dead and live loads.

Fig. 3 illustrates the limiting stress distribution under full service loading. The top (f_{1r}) and bottom (f_{2r}) stress ranges are the stress increments available to resist the superimposed loads. The superimposed dead and live load moments are defined as M_D and M_L. The required section moduli are thus given as:

$$S_1 \geq \frac{M_D + M_L}{f_{1r}} \quad (13)$$

$$S_2 \geq \frac{M_D + M_L}{f_{2r}} \quad (14)$$

With reference to Figs. 2 and 3, the top stress range, f_{1r}, is given as:

$$f_{1r} = f_{ti} - f_{cs} - \frac{\Delta M_o}{S_1} - \Delta_1 f_1 - \Delta_2 f_1 \quad (15)$$

or, upon substitution of Eqs. (6) and (11):

$$f_{1r} = Rf_{ti} - f_{cs} - (1 - R)\frac{M_{ot}}{S_1} - \frac{\Delta M_o}{S_1} - \frac{P_e}{S_1}(e_t - e_s) \quad (16)$$

The bottom stress range is given by:

$$f_{2r} = f_{ts} - f_{ci} - \frac{\Delta M_o}{S_2} - \Delta_1 f_2 - \Delta_2 f_2 \quad (17)$$

and upon substitution of Eqs. (7) and (12) becomes:

$$f_{2r} = f_{ts} - Rf'_{ci} - (1 - R)\frac{M_{ot}}{S_2} - \frac{\Delta M_o}{S_2} - \frac{P_e}{S_2}(e_t - e_s) \quad (18)$$

Eq. (16) is substituted into Eq. (13) and rearranged to produce the upper section modulus requirement:

$$S_1 \geq \frac{(1 - R)M_{ot} + \Delta M_o + M_D + M_L + P_e(e_t - e_s)}{Rf_{ti} - f_{cs}} \quad (19)$$

Eq. (14) is the lower section modulus requirement which, after substitution of Eq. (18) and rearranging becomes:

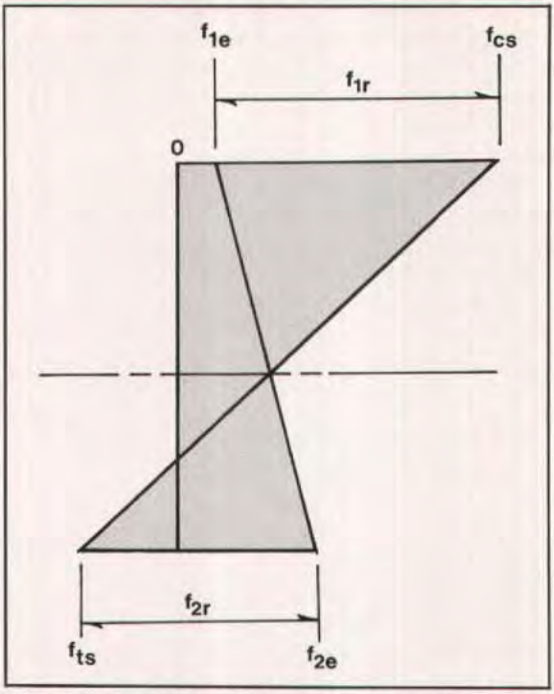

Fig. 3. Full service critical section stress distribution due to superimposed loads.

$$S_2 \geq \frac{(1-R)M_{ot} + \Delta M_o + M_D + M_L + P_e\,(e_t - e_s)}{f_{ts} - Rf_{ci}} \tag{20}$$

The above two equations, together with Eqs. (1), (2), and (4) are the generalized flexural design equations for prestressed concrete beams.

Similar equation sets are developed in Ref. 3; one set for the specific case wherein the locations of the two critical sections coincide, and a second set applicable to beams having a constant steel eccentricity. The equations developed herein reduce correctly to those in Ref. 3 when applied to those specific cases. For example, in beams having a constant steel eccentricity, the variable M_{ot} in Eqs. (19) and (20) is zero, ΔM_o then equals M_{os}, and the quantity $P_e\,(e_t - e_s)$ is zero as well. Eqs. (19) and (20) reduce to:

$$S_1 \geq \frac{M_{os} + M_D + M_L}{Rf_{ti} - f_{cs}} \tag{21}$$

and

$$S_2 \geq \frac{M_{os} + M_D + M_L}{f_{ts} - Rf_{ci}} \tag{22}$$

Eqs. (1) and (2) are unaffected by critical section locations or steel profile, and the last term in Eq. (4) is zero. Eqs. (1), (2), (4), (21), and (22) are equivalent to those found in Ref. 3.

An estimate of the ratio c_1/h prior to section selection using Eqs. (19) and (20) is required only when P_e is needed in those two equations [thus requiring P_i from Eq. (2)]. However, for normal concrete strengths, effectiveness ratios, and ratios of concrete strength at transfer to the 28-day strength, the upper (S_1) and lower (S_2) section moduli requirements from Eqs. (19) and (20) will typically be similar in magnitude since the denominators of the two equations will have similar magnitudes.

A bisymmetrical section is then indicated, with the ratio c_1/h equal to 0.50. If the design equations are being used to select a standard section (a double tee section of appropriate depth, for example), a reasonable estimate of the ratio can be arrived at by examining listed section properties for the standard sections.

The design equations developed can be applied directly to general prismatic prestressed concrete flexural members, and provide a systematic approach to working stress design of such elements. Their development follows closely those presented in previous works, and are based on accepted principles of flexural behavior. Their use, however, cannot guarantee an acceptable final design in all cases.

The maximum eccentricity from Eq. (4) may exceed the maximum eccentricity allowed by geometric or cover constraints, and the equations do not ensure adequate deflection characteristics or flexural strength in all cases. These possibilities must be considered before the design can be considered acceptable.

NUMERICAL EXAMPLES

Two numerical examples are presented to illustrate the use of these equations (Note: 1 in. = 25.4 mm, 1 kip = 4.448 kN, 1 psi = 6.895 kN/m², 1 kip/ft = 14.59 kN/m, 1 ft-kip = 1.356 kN•m.)

EXAMPLE 1

A 60 ft simply supported pretensioned beam is to carry a concentrated live load, L, at one of the two third points only. There will be no superimposed dead load. The beam section is rectangular with a width of 16 in. and a total depth of 36 in. The self weight, w_o, of the beam is 0.60 kips per ft. The strands are to be draped at midspan, with the eccentricity reducing linearly to zero at the beam ends.

The specified concrete strength at transfer, f'_{ci} is 5000 psi, and the specified 28-day concrete strength, f'_c, is 6000 psi. An effectiveness ratio, R, of 0.80 will be assumed. The design equations will be used to determine the allowable magnitude of the service live load, L, the allowable midspan eccentricity, and the required initial prestressing force, P_i.

ACI allowable stresses will be used as follows:

$f_{ti} = 3\sqrt{f'_{ci}} = 212$ psi
$f_{ci} = 0.6\sqrt{f'_{ci}} = -3000$ psi
$f_{ts} = 6\sqrt{f'_c} = 464$ psi
$f_{cs} = 0.45 f'_c = 2700$ psi

The concrete section properties are:

$S_1 = S_2 = 3456$ in.³
$A_c = 576$ in.²
$c_1 = c_2 = 18.0$ in.

The beam is shown in Fig. 4. By inspection, the critical section at transfer is the midspan section. The critical section at full service load will almost certainly be at the third point under the load L. The maximum total bending moment will occur at either the third point or midspan (to be determined later), and the eccentricity at the third point is less than the midspan eccentricity.

The self weight moment at the critical section at transfer, M_{ot}, equals:

$$M_{ot} = \frac{w_o L^2}{8} = 270 \text{ kip-ft}$$

By Eq. (1), $f_{cg} = -1394$ psi
By Eq. (2), $P_i = 802{,}944$ lb and
$P_e = RP_i = 642{,}355$ lb
By Eq. (4), $(e_t)_{max} = 10.95$ in.

The eccentricity at the third point is thus:

$$e_s = \frac{2}{3} e_t = 7.30 \text{ in.}$$

The self weight moment at the third

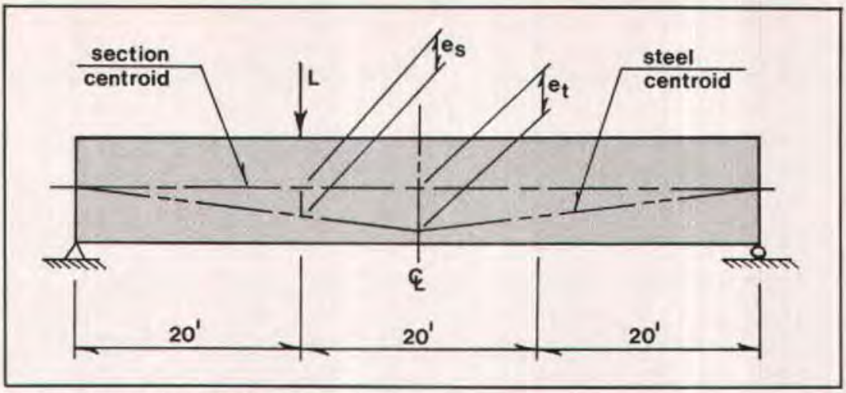

Fig. 4. Beam elevation for Example 1.

point, M_{os}, is given by:

$$M_{os} = \frac{w_o L^2}{9} = 240 \text{ kip-ft}$$

and

$$\Delta M_o = 240 - 270 = -30 \text{ kip-ft}$$

From Eq. (19), the maximum allowable live load moment is:

$$(M_L)_{max} = 607.06 \text{ kip-ft}$$

From Eq. (20), the maximum allowable live load moment is:

$$(M_L)_{max} = 605.74 \text{ kip-ft}$$

Thus, since the moment determined from Eq. (20) governs, the allowable load L at the third point is 45.43 kips. The total service load moment at the third point is 605.74 kip-ft plus the self weight moment of 240 kip-ft, or 845.74 kip-ft. The total moment at midspan is 724.31 kip-ft. Thus, the ratio of maximum total moment to prestress moment (eccentricity) is highest at the third point under the load, and the assumed critical section location at full service loading is correct.

Flexural stresses at the critical section at transfer are:

$$f_1 = 212.6 \text{ psi}, f_2 = -3000.6 \text{ psi}$$

These values are virtually identical to the allowable stresses. Flexural concrete stresses at the third point under the service load are:

$$f_1 = -2695.0 \text{ psi}, f_2 = 464.6 \text{ psi}$$

These values are, again, very near the allowable stress limits.

It should be noted, however, that the results will not in every case be quite so optimal. Sections actually selected in design will frequently have one or both section moduli exceeding the required values, and the actual stresses will hence not compare as well with the allowable stresses.

Further, the numerical values of the denominators of Eqs. (19) and (20) were virtually identical in this example, due to the values of R and the concrete strengths at transfer and full service. Thus, the section moduli required were also virtually identical, and a bisymmetric section is an appropriate choice.

EXAMPLE 2

A PCI standard 8 ft double tee section will be selected for a roof member given a span length of 36 ft, a superimposed dead load of 10 psf, and a live load of 25 psf. The beam elevation is shown in Fig. 5, and the cross section is shown in Fig. 6. Straight strands will be used, and a concrete topping will not be employed. An effectiveness ratio, R, of 0.85 will be assumed, and concrete with a release strength of 4000 psi and a design strength of 5000 psi will be used.

ACI Code allowable concrete stresses are used as follows:

$$f_{ti} = 6\sqrt{f'_{ci}} = 379 \text{ psi}$$

$$f_{ts} = 6\sqrt{f'_c} = 424 \text{ psi}$$

$$f_{ci} = 0.6 f'_{ci} = -2400 \text{ psi}$$

$$f'_{cs} = 0.45 f'_c = -2250 \text{ psi}$$

A beam weight, w_o, of 340 plf will be assumed. The minimum section moduli are determined from Eqs. (21) and (22) in which:

$$M_{os} = \frac{w_o L^2}{8} = 55.1 \text{ kip-ft}$$

$$w_D + w_L = (8)(35) = 280 \text{ plf}$$

$$M_D + M_L = \frac{(w_D + w_L)L^2}{8} = 45.4 \text{ kip-ft}$$

From Eq. (21): S_1 = 469. in.^{3} (minimum).

From Eq. (22): S_2 = 489. in.^{3} (minimum).

Several double tees and their section properties are shown in Table 1.

While all sections listed have an ade-

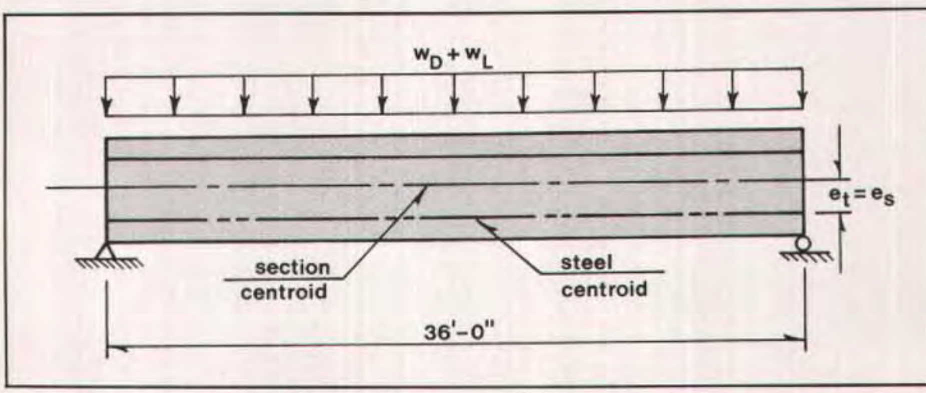

Fig. 5. Beam elevation for Example 2.

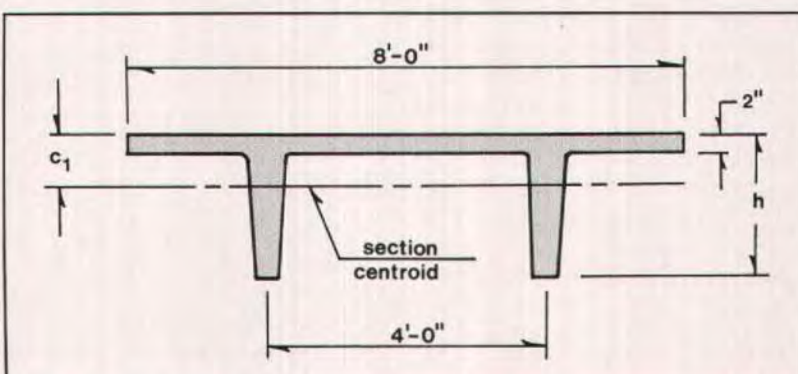

Fig. 6. Double tee beam cross section for Example 2.

Table 1. Summary of section properties for various double tees.

Designation	h, in.	A_c, in.^{2}	w_o, plf	S_1, in.^{3}	S_2, in.^{3}	c_1/h
8DT 14	14	306	319	1307	429	0.246
8DT 16	16	325	339	1630	556	0.254
8DT 18	18	344	358	1966	701	0.262

quate upper section modulus for this application, the 8DT 16 is the lightest section having an adequate lower section modulus. This section is an appropriate selection in this example. The self weight of 339 plf is close to the assumed value, and repeating the selection process with the correct value is unnecessary. From Eqs. (1), (2), and (4), the maximum permissible eccentricity (governed by the critical section at transfer, e.g., the beam end) is:

$(e_t)_{max} = 10.83$ in.

The minimum eccentricity, governed by the bottom fiber concrete stress at the full service load critical section (midspan) can be computed as:

$e_s = 9.03$ in.

The eccentricities above were found using a value for the prestress force at transfer (P_i) of 106,230 lb as determined from Eqs. (1) and (2). Note that a single acceptable value for the eccentricity was not found, but rather a range of values. This is due to the selected section having excess capacity, since both section moduli are greater than the minimum required. Required flexural strength, acceptable deflections, and adequate concrete cover over the strands must be determined prior to final acceptance of this design, as stated previously.

CONCLUSION

The equation set, when compared with previously developed sets, explicitly accounts for the differences in self weight moment and eccentricity between the critical sections at transfer and at full service loading. The equations are thus applicable to the design of noncomposite prestressed concrete beams regardless of loading and steel profile.

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APPENDIX — NOTATION

A_c = cross-sectional area of beam, in.^{2}

c_1 = distance from neutral axis to top of beam, in.

c_2 = distance from neutral axis to bottom of beam, in.

e_s = steel centroid eccentricity at full service load critical section, in.

e_t = steel centroid eccentricity at critical section at prestress force transfer, in.

f'_c = specified 28 day compressive strength of concrete

f'_{ci} = specified compressive strength of concrete at time of initial prestress

f_{ct} = allowable concrete compressive stress at transfer, psi

f_{cg} = concrete stress at neutral axis at transfer, psi

f_{cs} = allowable concrete compressive stress at full service, psi

f_{tt} = allowable concrete tensile stress at transfer, psi

f_{ts} = allowable concrete tensile stress at full service, psi

f_1 = top fiber concrete stress, psi

f_2 = bottom fiber concrete stress, psi

f_{1e} = top fiber concrete stress at full service critical section prior to application of superimposed loads, psi

f_{2e} = bottom fiber concrete stress at full service critical section prior to application of superimposed loads, psi

f_{1p} = top fiber concrete stress due to prestressing alone, psi

f_{2p} = bottom fiber concrete stress due to prestressing alone, psi

f_{1r} = top fiber concrete stress range available to superimposed loads, psi

f_{2r} = bottom fiber concrete stress range available to superimposed loads, psi

h = section depth, in.

L = span length, feet, and concentrated live load, kips, in Example 1

M_{os} = self weight moment at full service critical section, lb-in.

M_{ot} = self weight moment at critical section at transfer, lb-in.

M_D = superimposed dead load moment at full service critical section, lb-in.

M_L = superimposed live load moment at full service critical section, lb-in.

P_e = prestress force after losses, lb

P_t = prestress force immediately after force transfer, lb

R = effectiveness ratio, P_e/P_t

S_1 = upper section modulus, in.^{3}

S_2 = lower section modulus, in.^{3}

w_D = superimposed dead load, plf

w_L = superimposed live load, plf

w_o = beam weight, plf

ΔM_o = change in self weight moment from full service critical section to critical section at transfer, lb-in.

$\Delta_1 f_1$ = top concrete stress change from critical section at transfer to critical section at full service load, due to change in eccentricity between two sections, psi

$\Delta_1 f_2$ = bottom concrete stress change from critical section at transfer to critical section at full service load, due to change in eccentricity between two sections, psi

$\Delta_2 f_1$ = change in top fiber concrete stress due to time-dependent prestress losses, psi

$\Delta_2 f_2$ = change in bottom fiber concrete stress due to time-dependent prestress losses, psi

NOTE: Discussion of this paper is invited. Please submit your comments to PCI Headquarters by November 1, 1985.