Direct Minimum Weight Design of Long Span Prestressed Concrete Members

A. S. Prasada Rao
Scientist
Structural Engineering Research Centre
Madras, India

The elastic design of prestressed concrete members is based on two main assumptions: (1) sections which were plane before deformation remain plane after deformation and (2) strains remain proportional to stress.

With a material like concrete, these two assumptions remain satisfactory so long as the stresses do not exceed certain limits which depend upon the strength of concrete. Both assumptions become progressively more inexact as the stresses increase beyond these limits. However, in our present state of knowledge, the elastic theory of design still remains one of the most effective methods for proportioning a prestressed concrete member. The stress limits are generally chosen so that the structure will be usable and safe under service load conditions while ultimate load conditions are satisfied and a reasonably economical use is made of the material.

Prestressed concrete beams are generally classified as either short span or long span beams.[1] In long span beams, the self weight moment constitutes a major portion of the total moment. In all such cases, it is impractical to accommodate the eccentricity within the beam section given by the expression:

$$e = k_b - \frac{f_{ttp} Z_t}{P_t} + \frac{M_{sw}}{P_t}$$

where

k_b = bottom kern distance
f_{ttp} = permissible stress at top fiber of section at transfer
Z_t = section modulus with reference to top fiber of section
M_{sw} = self weight bending moment of member
P_t = prestressing force at transfer

The scope of this article is restricted to the design method for beams coming under the category defined by the above equation.

Synopsis

Describes a direct method (with the aid of a computer program) by which a minimum weight section can be obtained for long span prestressed concrete members without resorting to the use of either iterative techniques or design curves. The cross section derived by this procedure satisfies the transfer and working load permissible stresses at the bottom fiber, the working load permissible stress at the top fiber, and the cover requirements of the center of gravity of the prestressed reinforcement. It is shown that for either the given or assumed data the area of the section so worked out is the absolute minimum. Design guidelines are given for obtaining the least weight section of a member and a fully worked design example is also included.

LITERATURE REVIEW

Many attempts have been made in the past to obtain a rapid solution to the problem of determining the minimum weight section of a long span beam.[1-5]

An early classical contribution was that of Guyon.[1] His approach to the problem was to establish a balanced interaction between refined mathematical theory and actual practice. In his method, the effect of prestress losses is assumed initially to be negligible.

To account for this assumption, lower permissible stresses at the transfer stage than what otherwise would have been allowed in design were suggested. These assumptions enabled him to obtain an expression for the bottom fiber section modulus independent of the unknown self weight moment M_{sw}. By fixing the eccentricity of the prestressing force based on practical considerations, a quadratic expression in y_t involving the efficiency factor ρ_G can be obtained. At this stage a suitable value for ρ_G is assumed and solved for y_t. Knowing y_t and Z_b, the sectional dimensions b_t and b_b can be determined for the assumed values of d, t_b, t_t, and b_w.

For this improved section, the actual efficiency factor ρ_G is calculated. If the value is not close to the one assumed, a fresh value of ρ_G closer to the one obtained in the preceding cycle of iteration is assumed and the quadratic expression in y_t is again solved and fresh section dimensions are calculated. This procedure is continued until the assumed and calculated values of ρ_G are close to each other. The effective prestressing force is obtained from the permissible stress distribution at working load.

The shortcomings of this approach are (1) the method is essentially iterative and time consuming; (2) in determining the sectional dimensions, an unrealistic assumption, namely, that the effect of prestress losses is negligible, is initially made; and (3) the increased stress values in the section at transfer which occur because of the above assumption are either accepted or the section must be revised.

Another iterative method developed by Chetty, Prasada Rao, and Bhargava[3] makes use of Guyon's intuitive ideas described in Ref. 1. The effect of the loss of prestress is duly taken into account by introducing a loss factor η into the relevant expressions. The actual permissible stresses at transfer of prestress are used. The self weight moment M_{sw} is expressed in terms of the area of the section. The rest of the procedure is identical to that given in Ref. 1. This method is an improvement over Guyon's method since it does not make the unrealistic assumption that the effect of loss of prestress was negligible. However, it has the other drawback of being an iterative method.

Khachaturian, Ali and Thorpe[2] have developed an interesting method involving dimensionless parameters. In this method, the shape factor Δ_e that exactly satisfies the stresses at transfer and working load conditions is first calculated. A value of $\Delta > \Delta_e$ is assumed as a target. An expression involving the sectional area as an unknown is obtained in terms of Δ, ρ_k, η and the permissible stresses f_{bwp} and f_{btp} at the bottom fiber of the section. The factor ρ_k is the only unknown in the expression so obtained. For the assumed ratios of b_b/b_t and t/d, the ratio of b_w/b_t is obtained from the expression of the shape factor Δ. For this assumed value of Δ, the efficiency factor ρ_k is found from the relevant expression. Then A, b_t, and b_b are calculated. The eccentricity is determined from the cover requirements and the prestressing force is then computed by substituting this value of e in the bottom fiber equation of the working load condition.

The shortcomings of this method are: (1) an initial estimate of the shape factor Δ has to be made. This, in some cases, leads to a section that either gives too much or insufficient cover to the prestressing steel, so a minimum weight section is not obtained; and (2) the method assumes that the bottom and top flange thicknesses are identical. In many cases, a larger bottom flange thickness may be desirable to facilitate proper placement of tendons.

Nilson[5] has proposed another interesting iterative method for the design of long span beams. In this method, the self weight moment M_{sw} is divided into two parts. The first part comprises that portion of self weight moment $(1 - \mu)$ M_{sw} which can be treated as for any short span beam. The second part entails the balance of the self weight moment μM_{sw} which can be treated as additional applied moment. The maximum eccentricity is determined from cover requirements. This eccentricity is then equated to the one corresponding to the top fiber stress at transfer when the fictitious first part of the self weight bending moment $(1 - \mu)M_{sw}$ is assumed to be acting and solved for μ. If this value of μ differs from that assumed in calculating the section moduli, a revised value of μ is adopted. The procedure is continued until the assumed value of μ is equal to the one given by the eccentricity equation. The rest of the design procedure is identical to that of short span beams.

The shortcomings of this method are: (1) the approach is iterative and time consuming; (2) it does not lead to a minimum weight design; (3) the method does not directly lead to sectional dimensions; and (4) the resulting design may require additional prestressing force.

The method suggested by Saether[6] assumes initial values of d, b_w, t_b, t_t, and M_{sw}. The self weight required is checked with the initial assumed value. The rest of the procedure is similar to that given in Ref. 1, which is basically iterative. This method does not apply for long span beams because the calculated eccentricity either falls outside the section or gives insufficient cover.

Another direct method for the minimum weight design of prestressed concrete members was developed by Prasada Rao.[7] For a given beam depth, web width, and flange thicknesses, this method directly gives the section dimensions without any trial and error procedures. The computed section exactly satisfies all the permissible stresses. The method utilizes the principles developed by Guyon,[1] Chetty,[3] and Saether.[6] However, it does not work satisfactorily for long span members for the same reasons as Saether's method.

Through a newly introduced load interaction concept, Hatcher[4] has obtained expressions somewhat similar to those of Khachaturian.[2] In this method too, Δ greater than Δ_e is assumed and hence the designs worked out by this method do not lead to minimum weight design unless the designer resorts to additional trials.

Some methods based on mathematical optimization techniques have been published such as those of Morris.[8] In general, these methods consider all the probable variables involved in a given problem and try to obtain an optimum solution for the required objective function at the expense of valuable computer time. The eight variables involved in an elastic optimization problem are well known. As Naaman rightly pointed out in a discussion,[4] there are seventy combinations of solutions for these unknowns, taking any four of these quantities as known values.

The influence of some of the design variables such as d, b_w, t_t, and t_b on the weight of the desired section is explained later on in this article. For a minimum weight design, the engineer obviously takes the largest depth that is practically permissible. Similarly, he attempts to obtain the lowest web width that is admissible and the smallest feasible flange thicknesses. In a minimum weight design, the above variables become inactive once the values are assigned as suggested above.

Another interesting method involving the parallelogram-shaped interaction diagram between (P_t/A) and $(P_t\,e/A\,d)$ was developed by Somayaji.[9] In this method, the shape factor Δ, the efficiency factor ρ_k, and eccentricity ratio (e/d) are assumed and the section moduli Z_b and Z_t required are found for the given permissible stresses. The eccentricity and prestress force P_t are found from the interaction diagrams. This method involves an interesting concept but it neither leads to minimum weight design nor directly to the sectional dimensions.

In summary, it is clear from a review of the literature that the existing methods for solving long span prestressed concrete beams are iterative and furthermore might not always lead to a minimum weight design.

PROPOSED METHOD

This article suggests a procedure for the direct minimum weight design of long span prestressed concrete members. The section evaluated exactly satisfies the permissible stresses, f_{btp}, f_{bwp}, and f_{twp}, plus also the cover requirements d'.

This is accomplished by making use of the relation between the known data (permissible stresses f_{btp}, f_{bwp}, f_{twp}, and effective cover d') and the unknown data (y_t and ρ_G). For assumed or given data, d, b_w, t_b, and t_t, the area of the section A that gives exactly the required section modulus Z_b for carrying moments M_A and M_{sw} is also obtained in terms of the unknown y_t.

While expressing the parameters of the cross section in terms of the unknown efficiency factor ρ_G, advantage is taken of the well-known relation between Z_b, A, ρ_G and y_t. The solution of the two expressions of ρ_G obtained from the procedure given above leads to a cubic equation in y_t, the solution of which when substituted in the corresponding expressions gives the values of ρ_G, A, and P_t. The derivations of the relevant expressions that lead to the direct design of minimum weight section are given on the following pages.

Basic Equations

The four basic equations that govern the design of a simply supported prestressed concrete member with varying tendon eccentricity are:

$$P_t/A - P_t e/Z_t + M_{sw}/Z_t = f_{tta} \geqslant f_{ttp} \qquad (1)$$

$$P_t/A + P_t e/Z_b - M_{sw}/Z_b = f_{bta} \leqslant f_{btp} \qquad (2)$$

$$\eta P_t/A - \eta P_t e/Z_t + M_{sw}/Z_t + M_A/Z_t = f_{twa} \leqslant f_{twp} \qquad (3)$$

$$\eta P_t/A + \eta P_t e/Z_b - M_{sw}/Z_b - M_A/Z_b = f_{bwa} \geqslant f_{bwp} \qquad (4)$$

The mathematical symbols in Eqs. (1) to (4) are defined in the Notation section (see Appendix) and are also graphically shown in Fig. 1.

Note that all the permissible stresses are assumed to be compressive and positive. If any of the stresses f_{ttp} or f_{bwp} are tensile, then a negative value should be substituted in the relevant expressions. It was shown by Guyon[1] that, for long span members, Inequality (1) would always be satisfied. In such cases, the eccentricity derived from practical considerations as given by Inequality (5a) is usually adopted:

$$e \leqslant y_b - d' \qquad (5a)$$

Guyon[1] and Khachaturian[2] have shown that the section which exactly satisfies the equations obtained after expressing Inequalities (2), (3), (4), and (5a) as equalities and which possesses a higher efficiency factor ρ_G or ρ_k would be the minimum weight section. The required expressions for finding directly the sectional dimensions of a minimum weight section are derived below.

For a simply supported member, the self weight moment M_{sw} is given by:

$$M_{sw} = C\,QL^2A \qquad (6)$$

The section modulus Z_b, A, and y_t are related as follows:

$$Z_b = \rho_G A y_t \qquad (7)$$

in which Guyon's efficiency factor ρ_G is defined by:

$$\rho_G = r^2/(y_b y_t) \qquad (8a)$$

By definition, the loss factor η is given by:

$$\eta = P_e/P_t \qquad (9)$$

The following expression can be easily obtained from Fig. 1(a):

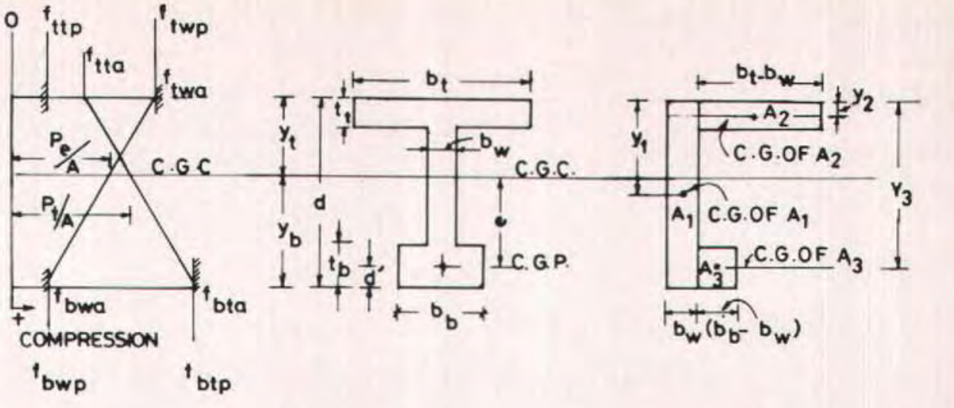
$$P_e/A = f_{twp}\, y_b/d + f_{bwp}\, y_t/d \qquad (10a)$$

Noting that $y_b = d - y_t$, Eq. (10a) can be written as:

$$P_e/A = f_{twp} - (f_{twp} - f_{bwp})\, y_t/d \qquad (10b)$$

The expression below can be deduced from Fig. 1b:

$$e = d - y_t - d' \qquad (5b)$$



a) STRESS DIAGRAM b) IDEALISED SECTION c) BREAK-UP OF IDEALISED SECTION

Fig. 1. Assumed beam section showing stress diagram and section breakdown.

Substituting Eqs. (6), (7), (9), (10b), and (5b), in Eq. (2), the following relation between the unknowns ρ_G and y_t in terms of the known values of permissible stresses, beam depth and the effective cover d' is obtained.

$$\rho_G = (C_1 y_t^2 + C_2 y_t + C_3)/(C_1 y_t^2 + g\, y_t) \tag{8b}$$

where

$$C_1 = f_{bwp} - f_{twp} \tag{11}$$

$$C_2 = d\, f_{twp} + (d - d')(f_{twp} - f_{bwp}) \tag{12}$$

$$C_3 = C\, QL^2\, \eta\, d - d\,(d - d')\, f_{twp} \tag{13}$$

and

$$g = d\,(f_{twp} - \eta\, f_{btp}) \tag{14}$$

From Eqs. (2) and (4), the following expression for the section modulus Z_b is obtained:

$$Z_b = [M_A + (1 - \eta)M_{sw}]/(\eta\, f_{btp} - f_{bwp}) \tag{15a}$$

Substituting for M_{sw} from Eq. (6) in Eq. (15a) yields:

$$Z_b = M_A/D + BA/D \tag{15b}$$

where

$$D = \eta\, f_{btp} - f_{bwp} \tag{16}$$

and

$$B = (1 - \eta) C\, QL^2 \tag{17}$$

The assumed section is given in Fig. 1b. Fig. 1c is a breakdown of the beam section given in Fig. 1b which facilitates the computation of section properties.

From Fig. 1c:

$$A = A_1 + A_2 + A_3 \tag{18a}$$

Taking static moment of areas about the top fiber of the beam section and equating it to the static moment of the total area of the section about the same fiber gives:

$$A_1 y_1 + A_2 y_2 + A_3 y_3 = (A_1 + A_2 + A_3) y_t \tag{19}$$

or

$$A_3 = [A_1 (y_t - y_1) + A_2 (y_t - y_2)]/(y_3 - y_t) \tag{20}$$

From Eqs. (18a) and (20):

$$A = [A_1 (y_3 - y_1) + A_2 (y_3 - y_2)]/(y_3 - y_t) \tag{18b}$$

The moment of inertia of the assumed section is given by:

$$I = A_1 \frac{d^2}{12} + A_2 \frac{t_t^2}{12} + A_3 \frac{t_b^2}{12} + A_1 y_1^2 + A_2 y_2^2 + A_3 y_3^2 - A y_t^2 \tag{21a}$$

Substituting for $(A y_t)$ and A_3 from Eqs. (19) and (20), respectively, the following equation is obtained:

$$I = A_1 \left[\frac{(d^2 + 12 y_1^2 - 12 y_1 y_t)(y_3 - y_t) + (t_b^2 + 12 y_3^2 - 12 y_3 y_t)(y_t - y_1)}{12 (y_3 - y_t)} \right] + A_2 \left[\frac{(t_t^2 + 12 y_2^2 - 12 y_2 y_t)(y_3 - y_t) + (t_b^2 + 12 y_3^2 - 12 y_3 y_t)(y_t - y_2)}{12 (y_3 - y_t)} \right] \tag{21b}$$

The moment of inertia should also be equal to the product of the bottom fiber section modulus and the bottom fiber distance from the section center of gravity:

$$I = Z_b y_b = Z_b (d - y_t) \tag{21c}$$

From Eqs. (15b), (18b), and (21c):

$$I = \left[\frac{M_A}{D} + \frac{B}{D} \left\{ \frac{A_1 (y_3 - y_1) + A_2 (y_3 - y_2)}{(y_3 - y_t)} \right\} \right] (d - y_t) \tag{21d}$$

Equating Eqs. (21b) and (21d), an expression for A_2 is obtained and substituting it into Eq. (18b) and simplifying, the following expression for the area of the section is obtained:

$$A = \frac{Z a_1 y_t^2 + [a_2 - Z a_1 (d + y_3)] y_t - [y_3 (a_2 - Z a_1 d)]}{a_1 y_t^3 + (\phi_1 - a_1 C_4) y_t^2 + [C_4 a_1 (d + y_3) - \phi_3] y_t + (y_3 \phi_5 - C_4 a_1 d y_3)} \tag{18c}$$

where

$$Z = M_A/D \tag{22}$$

$$a_1 = 12 (y_3 - y_2) \tag{23}$$

$$a_2 = A_1 [d^2 (y_3 - y_2) + t_t^2 (y_1 - y_3) + t_b^2 (y_2 - y_1) - a_1 (y_1 - y_3)(y_2 - y_1)] \tag{24}$$

$$\phi_1 = t_t^2 - t_b^2 - a_1 (2 y_3 + y_2) \tag{25}$$

$$\phi_3 = 2\, y_3\, t_t^2 - t_b^2\,(y_2 + y_3) - a_1\, y_3\,(2\, y_2 + y_3) \tag{26}$$

$$\phi_5 = t_t^2\, y_3 - t_b^2\, y_2 - y_3\, y_2\, a_1 \tag{27}$$

$$C_4 = B/D \tag{28}$$

From Eqs. (7) and (15b):

$$\rho_G = (M_A + B\,A)/(D\, y_t A) \tag{8c}$$

Substituting the formula for A given by Eq. (18c) in Eq. (8c) which expresses ρ_G and equating the resulting formula to Eq. (8b) which also expresses ρ_G and simplifying, the following cubic equation in y_t is obtained:

$$\begin{aligned}\{a_2 C_1 - Z\,(C_1\,\phi_1 + a_1\,\phi_2)\}\, y_t^3 &+ [a_2\{C_2 - C_1\,(y_3 + C_4)\} + Z(C_1\,\phi_3 - g\,\phi_1 - a_1\,\phi_4)]\, y_t^2 \\ &+ [a_2\{C_3 - C_4\,(g - C_1\, y_3) - y_3\, C_2\} + Z\,(g\,\phi_3 - C_1\, y_3\,\phi_5 - a_1\,\phi_6)]\, y_t \\ &+ [a_2\, y_3\,(C_4\, g - C_3) + Z\, y_3\,(a_1\, d\, C_3 - g\,\phi_5)] = 0\end{aligned} \tag{29}$$

where

$$\phi_2 = C_1\,(d + y_3) - C_2 + g \tag{30}$$

$$\phi_4 = C_2\,(d + y_3) - C_3 - C_1\, d\, y_3 \tag{31}$$

$$\phi_6 = C_3\,(d + y_3) - C_2\, d\, y_3 \tag{32}$$

The solution of the cubic Eq. (29) gives the value of y_t of the minimum weight section that also satisfies the cover requirements. The efficiency factor ρ_G is obtained by substituting the value of y_t obtained from Eq. (29) in Eq. 8(b).

Rearranging Eq. (8c):

$$A = M_A/(\rho_G\, D\, y_t - B) \tag{18d}$$

Substituting in Eq. (18d) the values of ρ_G and y_t derived above, the value of cross-sectional area A is obtained. Rearranging Eq. (18b):

$$A_2 = \frac{A\,(y_3 - y_t) - A_1\,(y_3 - y_1)}{(y_3 - y_2)} \tag{33}$$

Substituting the values of y_t and A in Eq. (33), the area A_2 is found. Rearranging Eq. (18a):

$$A_3 = A - (A_1 + A_2) \tag{34}$$

Knowing A, A_1, and A_2, A_3 can be obtained from Eq. (34). The flange widths can now be calculated:

$$b_t = b_w + \frac{A_2}{t_t} \tag{35}$$

$$b_b = b_w + \frac{A_3}{t_b} \tag{36}$$

The eccentricity e and the prestressing force P_e are found by substituting the value of y_t in Eqs. (5b) and (10b), respectively. The proportioning of the idealized section is now complete and the design details can be worked out.

Computer Program

Based on the proposed method a computer program was written in Fortran IV to suit a Prime computer. A simplified flow chart of the program is given in Fig. 2. Readers interested in obtaining a print-out of the program can get a copy from PCI Headquarters.

PRACTICAL GUIDELINES

In this method, the permissible stresses, span of beam, and applied loading (excluding self weight) are given. The loss factor, effective cover, beam depth, web width, top flange thickness, and the bottom flange thickness are assumed. For the above data (assumed as well as given), this method gives the minimum weight section of the beam. The various assumptions are discussed below.

Effective Cover d'

Experience indicates that the effective cover d' should be of the order of $0.10d$. It appears that the section that gives a greater d' than the minimum needed will be heavier than the section that gives exactly the required d'. However, if the heavier section is adopted, giving a larger d' than desired, some advantage can be taken by increasing the eccentricity to its practical limit and reducing the effective prestressing force. These aspects are further elaborated upon in the design example given later in this paper.

Beam Depth d

Frequently, the depth of a beam is governed by clearance considerations or architectural requirements. As a guide, the depth can be chosen to be about $(1/20) \times$ span for long span beams. This criterion, however, is only a rough estimate. If there are no restrictions, it is obviously advantageous from an economic viewpoint to choose a beam with a fairly large depth.

Loss Factor η

Prestress losses can be estimated from past experience. Under normal conditions, the loss factor is about 0.85 and 0.80 for post-tensioned and pretensioned concrete, respectively.

Web width b_w

It is desirable to adopt a beam section configuration that gives a relatively large value of ρ_G since an increase in ρ_G results in a decrease in the cross-sectional area of the member. This becomes apparent after studying Eqs. (7) and (8a). Consider the section given in Fig. 1(b). For discussion purposes, assume that the given section consists of two sets of rectangles, one comprising the web and the other the overhanging portions of both flanges.

The value of ρ_G can vary from 0.33 for a rectangular section to a value as high as 0.55 for a well-proportioned flanged section. If the thickness of the web of a flanged section is increased while keeping the total area of the section constant, the section efficiency ρ_G moves towards 0.33, i.e., the value corresponding to a rectangular section, because the influence of the web area increases. It follows, therefore, that a minimum practical web thickness should be adopted which can permit easy placement and compaction of concrete. Guyon's[1] guidelines are:

When $d < 30$ in. (76 cm),

$$b_w = d/7 \text{ to } d/8 \qquad (37)$$

When $d > 30$ in. (76 cm),

$$b_w = 4 \text{ in. } (10 \text{ cm}) + \frac{d}{40} \qquad (38)$$

If some tendons pass through the web, the following formula is suggested:

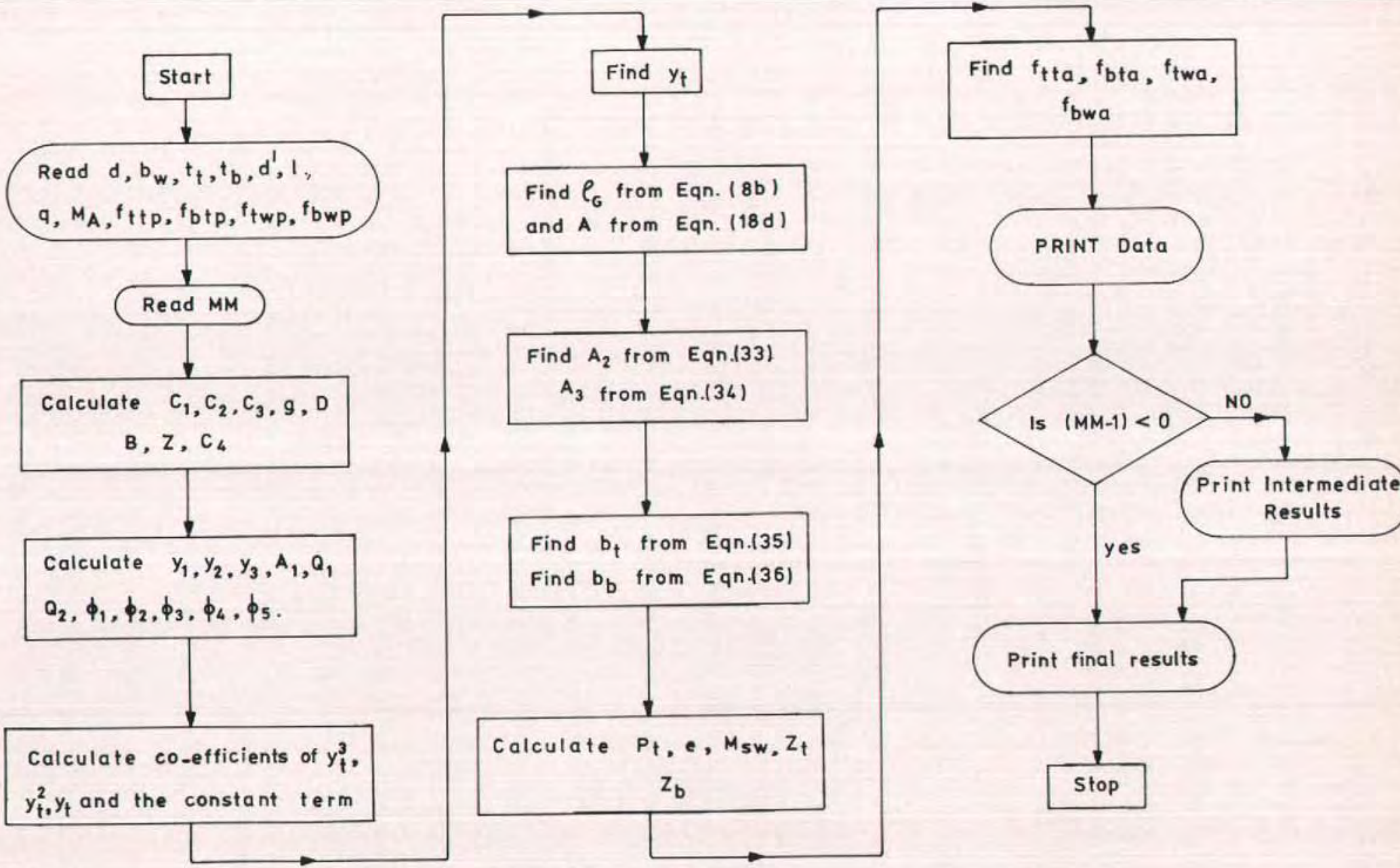


Fig. 2. Simplified flow chart of computer program.

$$b_w \geq \frac{d}{40} + 2 \text{ covers} + \Sigma \phi \qquad (39)$$

where ϕ is the diameter of the tendons located in the web.

Flange thickness, t_b and t_t

As was explained previously, the efficiency ρ_G of a beam section increases if a larger portion of the total cross-sectional area is placed in the flanges as compared to that of the web. The new flange area will also have a relatively large static moment. In such a case the efficiency factor ρ_G moves toward a value of 0.55. This is possible if the flange thicknesses are relatively thin.

The practical limits suggested for the top flange are between 0.10 d to 0.15 d. A larger bottom flange thickness is sometimes required for easy placement of prestressing tendons. This also reduces the bottom flange width. Note that the bottom flange thickness is usually about 0.20 d.

DESIGN EXAMPLE

DATA: A prestressed concrete beam (see Fig. 3) with the following properties and assumptions:

L = 75 ft or 900 in. (2286 cm)
f'_c = 6000 psi (4140 N/cm²)
$f_{ttp} = -150$ psi (-103.4 N/cm²)
f_{btp} = 2673 psi (1843 N/cm²)
$M_A = 19 \times 10^6$ (in.-lb) (214.51×10^6 N-cm)
f_{twp} = 1760 psi (1213.5 N/cm²)
$d \ngtr 50$ in. (127 cm)
$f_{bwp} = -150$ psi (-103.4 N/cm²)
$t_t \nless 7.5$ in. (19.05 cm)
$t_b \nless 9.0$ in. (22.86 cm)

REQUIRED: Find the minimum weight section of this beam while keeping the above constraints.

SOLUTION: Use the following eight steps together with the equations derived in the paper to solve the problem.

Step 1 — Select suitable values for d, t_t, t_b and b_w.

In accordance with the practical guidelines given in this article, the minimum weight section with the constraints stipulated in the given data is obtained by selecting:

d = 50 in. (127 cm), t_b = 9.0 in. (22.86 cm), t_t = 7.5 in. (19.05 cm)

The web width is the greater of:

$b_w \geq 50/40 + 4 = 5.25$ in. (13.34 cm)

Assume that one 12/5-mm tendon will pass through the web and that the diameter of the tendon is 1.417 in. (3.6 cm).

Then: $b_w \geq 50/40 + 2(1.417) + 1.417 = 5.50$ in. (13.97 cm)

Hence, adopt b_w = 5.50 in. (13.97 cm).

Assume the beam is post-tensioned.

Then $\eta = 0.85$ and $d' = 0.1\, d = 0.1 \times 50 = 5.0$ in. (12.70 cm)

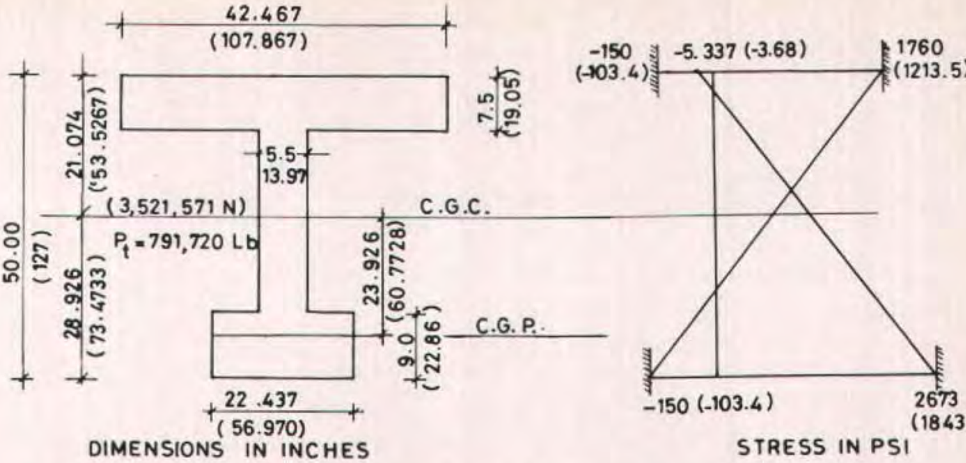


Fig. 3. Design example. Note: quantities in parentheses are in "cm" or "N/cm²."

Step 2 — Calculate the parameters C_1, C_2, C_3, g, D, B, Z, and C_4.

Substituting relevant values in Eqs. (11), (12), (13), (14), (16), (17), (22), and (28), respectively, the following values are obtained:

$C_1 = (-150 - 1760) = -1910$ psi $(-13.1695$ N/mm$^2)$

$C_2 = 50 \times 1760 + (50 - 5)[1760 - (-150)]$

$= 173{,}390$ lb/in. (304587 N/cm)

$C_3 = 1/8 \times 0.086806 \times 900^2 \times 0.85 \times 50 - 50(50 - 5)1760$

$= -3{,}586{,}466$ lb $(-15952.6 \times 10^3$ N)

$g = 50(1760 - 0.85 \times 2673)$

$= -25{,}603$ lb/in. $(-44831$ N/cm)

$D = 0.85 \times 2673 - (-150) = 2422.05$ psi (16.70 N/cm^2)

$B = (1 - 0.85)\, 1/8 \times 0.086806 \times 900^2$

$= 1318.36$ lb/in. (2308.45 N/cm)

$Z = 19 \times 10^6/2422.05 = 7844.60$ in.^{3} (128550 cm^3)

$C_4 = 1318.36/2422.05 = 0.54432$ in. (1.3826 cm)

Step 3 — Calculate the parameters a_1, a_2, ϕ_1, ϕ_2, ϕ_3, ϕ_4, ϕ_5, and ϕ_6.

Using Eqs. (23), (24), (25), (30), (26), (31), (27), and (32), respectively.

$y_1 = 0.5\,d = 0.5 \times 50 = 25$ in. (63.50 cm)

$y_2 = 0.5\,t_t = 0.5 \times 7.5 = 3.75$ in. (9.525 cm)

$y_3 = d - 0.5\,t_b = 50 - 0.5 \times 9.0 = 45.5$ in. (115.57 cm)

$A_1 = b_w d = 5.5 \times 50 = 275$ cm^2 (1774.3 cm^2)

$a_1 = 12(45.50 - 3.75) = 501$ in. (1272.54 cm)

$a_2 = 275\,\{50^2(45.50 - 3.75) + 7.5^2\,(25 - 45.50)$

$+ 9.0^2 \times (3.75 - 25) - 501\,(25 - 45.50) \times (3.75 - 25)\}$

$= -321.06 \times 10^5$ in.^{5} $(-33943.2 \times 10^5$ cm$^5)$

$\phi_1 = 7.5^2 - 9.0^2 - 501\,(2 \times 45.50 + 3.75)$

$= 47{,}494.5$ in.^{2} $(-306435$ cm$^2)$

$\phi_2 = -1910\,(50 + 45.50) - 173950 - 25603$

$= -381{,}957.5$ lb/in. $(-668808$ N/cm)

$\phi_3 = 2x45.5 \times 7.5^2 - 9^2 (3.75 + 45.50) - 501x45.50 (2x3.75 + 45.50)$
$= -120.70 \times 10^4$ in.^{3} (-1977.91×10^4 cm^3)
$\phi_4 = 173950(50 + 45.50) + 3586466 + 1910 \times 45.50 \times 50$
$= 245.44 \times 10^5$ lb (1091.72×10^5 N)
$\phi_5 = 7.5^2 \times 45.5 - 9^2 \times 3.75 - 45.5 \times 3.75 \times 50$
$= -83{,}227.5$ in.^{3} (-136.385×10^4 cm^3)
$\phi_6 = -3586466 (50 + 45.5) - 173{,}950 \times 50 \times 45.50$
$= -738.24 \times 10^6$ lb-in. (-8342×10^6 N-cm)

Step 4 — Calculate the coefficients of the cubic Eq. (29)

Coefficient of $y_t^3 = -321.06 \times 10^5(-1910)$
$- 784.69 \{-1910(-47494.5) + 501(-381{,}957.5)\}$
$= 850.851 \times 10^6$ lb-in.^{3} (62034×10^9 N-cm^3)

Coefficient of $y_t^2 = -321.06 \times 10^5 \{173{,}950 + 1910 (45.50 + 0.54432)\}$
$+ 7844.60 (1910x120.70x10^4 - 25603x47494.5 -$
$501 \times 245.44 \times 10^5)$
$= -963.231 \times 10^{11}$ lb-in.^{4} (-178365×10^{11} N-cm^4)

Coefficient of $y_t = 321.06 \times 10^5 \{-3586466 - 0.54432(-25603 + 1910$
$\times 45.50)$
$- 45.50 \times 173950\} + 7844.60 (25603 \times 120.70x10^4$
$- 1910 \times 45.50 \times 83227.5 + 501 \times 738.24 \times 10^6)$
$= 345.741 \times 10^{13}$ lb.-in.^{5} (162616×10^{23} N-cm^5)

Constant term $= -321.06 \times 10^5 \times 45.50 (-0.54432 \times 1910 + 3586466)$
$+ 7844.60 \times 45.50 (-501 \times 50 \times 3586466 - 25603 \times 83227.5)$
$= -380.462 \times 10^{14}$ lb-in.^{6} (-454524×10^{14} N-cm^6)

Hence, the cubic equation that defines the center of gravity position of the minimum weight section is:

$850.851x10^9 y_t^3 - 963.231x10^{11} y_t^2 + 345.741x10^{13} y_t - 380.462x10^{14} = 0$
$y_t^3 - 113.208 y_t^2 + 4063.473 y_t - 44{,}715.47 = 0$

Step. 5 — Find the solution of the cubic equation.

The solution of this equation can be easily found by the iterative technique using the remainder theorem:

$$\begin{pmatrix}\text{Revised value}\\ \text{of } y_t\end{pmatrix} = \begin{pmatrix}\text{Assumed value}\\ \text{of } y_t\end{pmatrix} - \left(\frac{\text{First Remainder}}{\text{Second Remainder}}\right)$$

The solution is:
$y_t = 21.0737$ in. (53.5267 cm)
$y_b = 50 - 21.0737 = 28.9263$ in. (73.4733 cm)

Step 6—Find the values of ρ_G, A, A_2, A_3, b_t, and b_b from Eqs. (8b), (18d), (33), (34), (35), and (36), respectively.

$$\rho_G = \frac{-1910 \times 21.0737^2 + 173950 \times 21.0737 - 3586466}{-1910 \times 21.0737^2 - 25603 \times 21.0737}$$
$$= 0.5541$$

$$A = \frac{19000 \times 10^3}{0.5541 \times 2422.05 \times 21.0737 - 1318.36} = 704.683 \text{ in.}^2 \ (4546.615 \text{ cm}^2)$$

$$A_2 = \frac{704.683(45.50 - 21.0737) - 275(45.50 - 25)}{(45.50 - 3.75)} = 277.253 \text{ in.}^2 \ (1788.73 \text{ cm}^2)$$

$$A_3 = 704.683 - (275 + 277.253) = 152.430 \text{ in.}^2 \ (983.417 \text{ cm}^2)$$

$$b_t = 5.50 + \frac{277.253}{7.5} = 42.467 \text{ in.} \ (107.867 \text{ cm})$$

$$b_b = 5.5 + \frac{152.430}{9.0} = 22.437 \text{ in.} \ (56.990 \text{ cm})$$

Step 7—Calculate P_t, e, and M_{sw} from Eqs. (10a), (5a), and (6), respectively.

$$P_t = \frac{704.683}{0.85 \text{x} 50}(1760 \times 28.9263 - 150 \times 21.0737) = 791{,}720 \text{ lb} \ (3521571 \text{ N})$$

$$e = 28.9263 - 5.0 = 23.9263 \text{ in.} \ (60.7728 \text{ cm})$$

$$M_{sw} = 1/8 \times 0.086806 \times 900^2 \times 704.683 = 6193496 \text{ lb-in.} \ (6998.7 \times 10^4 \text{ N-cm})$$

Step 8—Calculate Z_b, Z_t, f_{tta}, f_{bta}, f_{twa}, and f_{bwa} from Eqs. (7), $(\rho_G A y_b)$, (1), (2), (3), and (4), respectively.

$$Z_b = 0.5541 \times 704.683 \times 21.0737 = 8204.092 \text{ in.}^3 \ (134441 \text{ cm}^3)$$

$$Z_t = 0.5541 \times 704.683 \times 28.9263 = 9941.990 \text{ in.}^3 \ (162919 \text{ cm}^3)$$

$$f_{tta} = \frac{791{,}720}{704.683} - \frac{791720 \times 23.9263}{9941.990} + \frac{6193496}{9941.99}$$

$$= 1123.512 - 1677.227 + 548.378 = -5.357 < -150 \text{ psi}$$

$$(-3.68 \text{ N/cm}^2) < (-103.4 \text{ N/cm}^2)$$

$$f_{bta} = \frac{791720}{704.683} + \frac{791720 \times 23.9263}{8204.092} - \frac{6193496}{8204.092}$$

$$= 1123.512 + 2302.206 - 752.719$$

$$= 2672.99 = 2673 \text{ psi} \ (1843 \text{ N/cm}^2)$$

$$f_{twa} = 0.85\,(1123.512 - 1677.227) + 548.378 + \frac{19 \text{x} 10^6}{9941.99}$$

$$= 954.985 - 1425.643 + 548.378 + 1682.279$$

$$= 1759.999 = 1760 \text{ psi} \ (1213.5 \text{ N/cm}^2)$$

$$f_{bwa} = 0.85\,(1123.512 + 2302.206) - 752.719 - \frac{19 \text{x} 10^6}{8204.092}$$

$$= 954.985 + 1956.875 - 752.719 - 2309.143$$

$$= -150.002 = -150.0 \text{ psi} \ (-103.4 \text{ N/cm}^2)$$

DISCUSSION OF MINIMUM WEIGHT DESIGN

The design example indicates the various steps involved in determining the direct minimum weight of long span prestressed concrete members. Following this method, different designs were evaluated by varying the effective cover d' and keeping the other input data unaltered. From these results curves were drawn as shown in Fig. 4 depicting the variation of A and P_t versus different values of d'.

It can be seen from Fig. 4 that as the value of d' increases, the sectional area also increases. As explained earlier, the practical value of d' is about $0.1d$ which works out to 5 in. (12.7 cm) in this example. The sections that give $d' < 5$ in. (12.7 cm) are therefore not practically admissible. The choice hence lies with sections that give $d' = 5$ in. (12.7 cm) and higher. It is therefore clear that the minimum weight section S_1, corresponding to $d' = 5$ in. (12.7 cm), has the least area. The prestressing force P_t of the section S_1 with $d' = 5$ in. (12.7 cm) is 791,720 lb (3513140 N). This combination exactly satisfies the stresses f_{twp}, f_{bwp}, and f_{btp}.

In practice, sections sometimes result in a value of d' greater than that required. For example, assume that S_2 is such a section (see Fig. 4). This section gives $d' = 7.5$ in. (19 cm), which is greater than the required value of 5 in. (12.7 cm). By adopting such a solution, some savings in the prestressing force can be realized. The prestressing force would be 790,996 lb (3519950 N) as shown in Fig. 5. In this solution the stresses f_{btp} and f_{bwp} are exactly satisfied

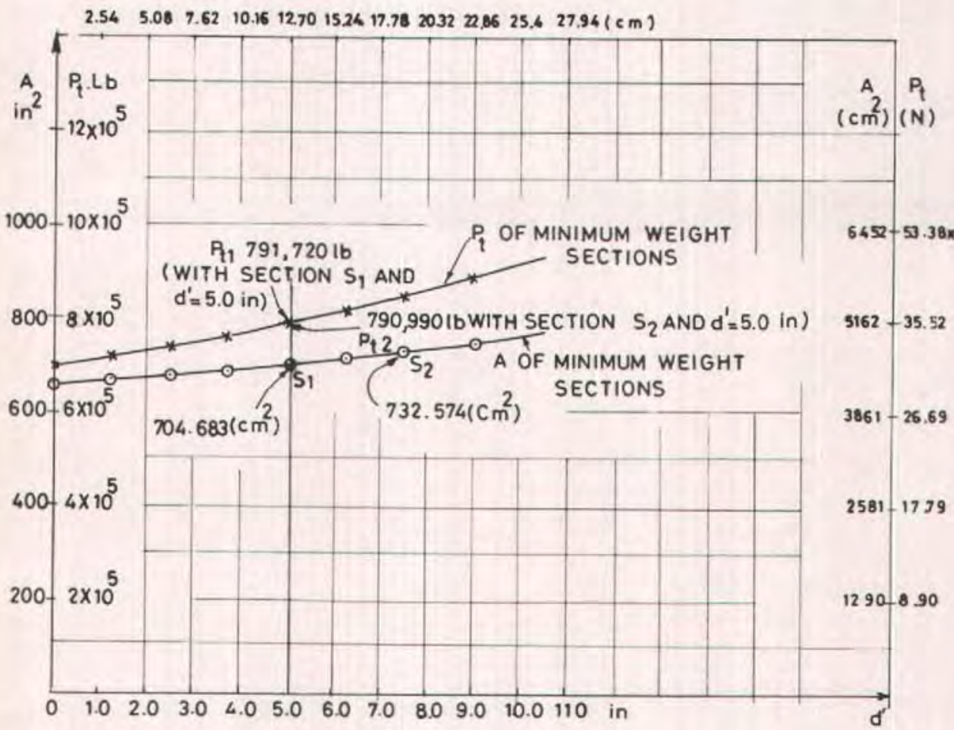


Fig. 4. Variation of A and P_t versus d' for minimum weight sections.

while f_{tta} and f_{twa} are within the permissible limits.

For a given section S_2, a combination of P_t and e can be found to satisfy the stress f_{twp} in addition to f_{btp} and f_{bwp}. Their values would be 850,006 lb

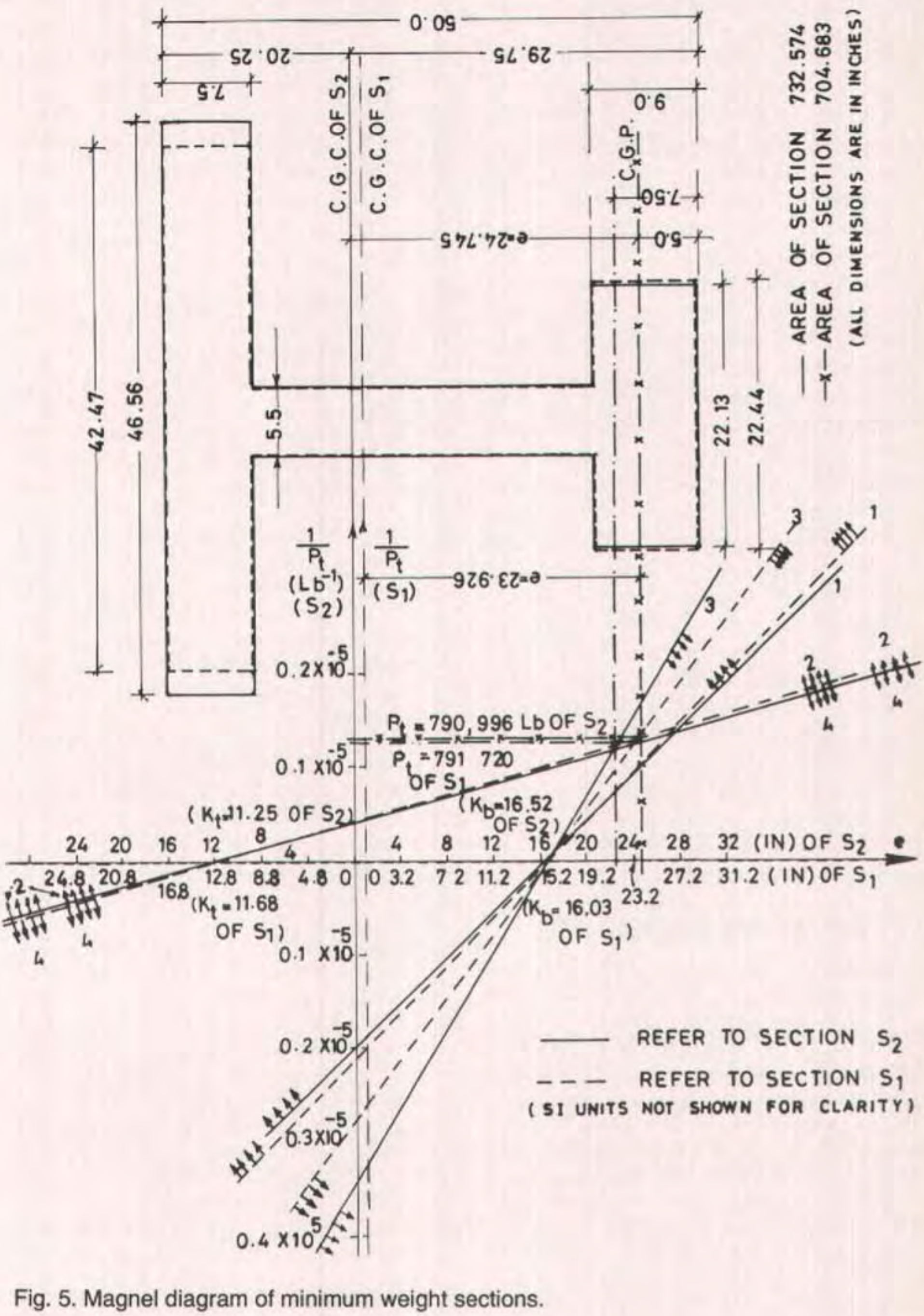


Fig. 5. Magnel diagram of minimum weight sections.

(3780000 N) and 22.495 in. (57.1 cm), respectively. It may be noted that an eccentricity of 22.495 in. (57.1 cm) gives a d' value of 7.5 in. (19 cm). The prestressing force could also be reduced by adopting an eccentricity corresponding to $d' = 5.0$ in. (12.7 cm) even though the section used is S_2.

The above discussion therefore compares two cases, namely, one that selects the section S_1, prestressing force P_t, and eccentricity e corresponding to $d' = 0.1\ d = 5.0$ in. (12.7 cm) while the other chooses the section S_2, prestressing force P_t, and eccentricity e corresponding to $d' = 0.1\ d = 5$ in. (12.7 cm). By choosing the first solution a savings of 3.81 percent in concrete area can be realized while the increase in prestressing force would be of the order of 0.092 percent. Therefore, a significant cost savings would accrue through the adoption of a minimum weight section.

The Magnel diagrams for the sections S_1 and S_2 are shown in Fig. 4. The cross-sectional details of the sections S_1 and S_2 are also shown in the same diagram. The Y-axes of the Magnel diagrams which show the variation of $1/P_t$ are made to coincide with the respective centroidal axes of the sections S_1 and S_2. This enables the designer to visualize whether or not the eccentricities of the prestressing forces corresponding to sections S_1 and S_2 would fall inside the section with required cover d'.

ACKNOWLEDGMENTS

The author wishes to express his appreciation to his colleagues Dr. B. Venkateswarlu, Shri V. S. Parameswaran and Dr. N. Lakshmanan for their valuable comments. He also wishes to thank his other colleague Shri G. Shanmugam for his help in computerizing this method. He expresses his indebtedness to the Director, Dr. M. Ramaiah and to Senior Deputy Director Shri N. V. Raman for ther constant guidance and encouragement throughout the preparation of this article. This paper is being published with the permission of the Director, Structural Engineering Research Centre, Madras, India.

CONCLUSION

The method given in this article determines directly the minimum weight section of long span prestressed concrete members. To save time the method has been computerized. Though the solution may require a somewhat higher prestressing force than other methods, this is offset by a relatively large reduction in the volume of concrete. This reduced weight will in turn bring savings in erection and transportation costs.

REFERENCES

1. Guyon, Y., *Prestressed Concrete*, V. 1.1, Asia Publishing House, Bombay, 1963, pp. 235-259.
2. Khachaturian, N., and Gurfinkel, G., *Prestressed Concrete*, McGraw Hill Book Company, New York, N.Y., 1969, pp. 212-252.
3. Chetty, S. M. K., Prasada Rao, A. S., and Bhargava, R. N., "Selection and Design of Prestressed Concrete Sections for Flexure," Reprint No. 61, *The Indian Concrete Journal*, Bombay, India, pp. 1-34.
4. Hatcher, David S., "Direct Flexural Design of Prestressed Concrete," *Journal of Structural Division*, ASCE, August 1976, pp. 1231-1249. Discussion by Naaman, A. E., *Journal of Structural Division*, ASCE, June 1979; Discussion by Prasada Rao, A. S., *Journal of Structural Division*, ASCE, May 1980, pp. 1222-1223.
5. Nilson, A. H., *Design of Prestressed Concrete*, John Wiley and Sons, New York, N.Y., pp. 131-134.
6. Saether K., "Direct Design of Prestressed Concrete Members," *ACI Journal*, February 1963, pp. 239-242.
7. Prasada Rao, A. S., "A Direct Method of Design of Prestressed Concrete Sections for Flexure," *Cement & Concrete*, India, October-December 1967, pp. 186-200.
8. Morris, David, "Prestressed Concrete Design by Linear Programming," *Journal of Structural Division*, ASCE, March 1978, pp. 439-452.
9. Somayaji, Shan, "Prestressed Concrete Flexural Member: Design," *Journal of Structural Division*, ASCE, August 1982, pp. 1781-1796.

APPENDIX — NOTATION

A = total cross-sectional area of section
A_1 = cross-sectional area of web for full depth of section $= b_w d$
A_2 = cross-sectional area of overhanging portions of top flange $= (b_t - b_w) t_t$
A_3 = cross-sectional area of overhanging portions of bottom flange $= (b_b - b_w) t_b$
a_1 $= 12 (y_3 - y_2)$
a_2 $= A_1 \{ d^2 (y_3 - y_2) + t_t^2 (y_1 - y_3) + t_b^2 (y_2 - y_1) - a_1 (y_1 - y_3) \times (y_2 - y_1) \}$
B $= (1 - \eta) CQL^2$
b_b = width of bottom flange
b_t = width of top flange
b_w = width of web
C = constant depending on support conditions (one-eighth for simply supported condition)
CGC = center of gravity of cross section
CGP = center of gravity of prestressing force
C_1 $= f_{bwp} - f_{twp}$
C_2 $= d f_{twp} + (d - d')(f_{twp} - f_{bwp})$
C_3 $= CQL^2 \eta d - d (d - d') f_{twp}$
C_4 $= B/D$
D $= \eta f_{btp} - f_{bwp}$
d = depth of cross section
d' = distance from beam soffit to CGS
E_t = modulus of elasticity of concrete at transfer
e = eccentricity of prestressing force
f'_c = specified compressive strength of concrete
f_{bta} = actual stress at bottom fiber of section at transfer
f_{bwa} = actual stress at bottom fiber of section at working load
f_{tta} = actual stress at top fiber of section at transfer
f_{twa} = actual stress at top fiber of section at working load
f_{btp} = permissible stress at bottom fiber of section at transfer
f_{bwp} = permissible stress at bottom fiber of section at working load
f_{ttp} = permissible stress at top fiber of section at transfer
f_{twp} = permissible stress at top fiber of section at working load
g $= d (f_{twp} - \eta f_{btp})$
k_b = bottom kern distance $= Z_t/A = r^2/y_t$
k_t = top kern distance $= Z_b/A = r^2/y_b$
L = span of beam
M_A = applied bending moment (excluding self weight bending moment)
M_{sw} = self weight bending moment
P_e = effective prestressing force
P_t = prestressing force at transfer
Q = unit weight of concrete
r = radius of gyration of section
t_b = bottom flange thickness
t_t = top flange thickness
y_b = distance of bottom fiber from CGC
y_t = distance of top fiber from CGC
y_1 $= 0.5\, d$
y_2 $= 0.5\, t_t$
y_3 $= d - 0.5\, t_b$
Z $= M_A/D$
Z_b = section modulus with reference to bottom fiber of section
Z_t = section modulus with reference to top fiber of section
Δ $= y_b/y_t$
Δ_e $= (\eta f_{btp} - f_{bwp})/(-\eta f_{ttp} + f_{twp})$
η = loss ratio P_e/P_t
ϕ = diameter of tendons located in web
ϕ_1 $= t_t^2 - t_b^2 - a_1 (2y_3 + y_2)$
ϕ_2 $= C_1 (d + y_3) - C_2 + g$
ϕ_3 $= 2 y_3 t_t^2 - t_b^2 (y_2 + y_3) - a_1 y_3 (2 y_2 + y_3)$
ϕ_4 $= C_2 (d + y_3) - C_3 - C_1 d y_3$
ϕ_5 $= t_t^2 y_3 - t_b y_2 - y_3 y_2 a_1$
ϕ_6 $= C_3 (d + y_3) - C_2 d y_3$
ρ_G = Guyon's efficiency factor $= r^2/(y_b y_t)$
ρ_K = Khachaturian's efficiency factor $= r^2/d^2$

* * *