David Darwin
Assistant Professor of Civil Engineering
University of Kansas
Lawrence, Kansas

Shear Component of Prestress by Equivalent Loads

The design of prestressed concrete beams for shear using the ACI Building Code[1] requires the calculation of the vertical or shear component of the effective prestress, V_p.

In most cases, the tendon profile is flat enough so that V_p $(= P \sin \alpha)$ may be approximated by $P \tan \alpha = P\, dy/dx$ (Fig. 1), where P equals the effective prestress force and α equals the angle made by the centroid of the prestressing steel and the horizontal.

In the past, the common design procedure has been to calculate dy/dx, the slope of the tendon, at points along the beam. While this procedure has proved satisfactory, it is tedious and ignores the effects of induced reactions in continuous beams. An alternative approach, presented here, uses the concept of "equivalent loading" to obtain the shear component of the prestress.

The equivalent loading concept has been used successfully for over two decades for flexural design[2] and is an integral part of the load balancing technique.[3]

While this technique is used by many structural engineers, the applica-

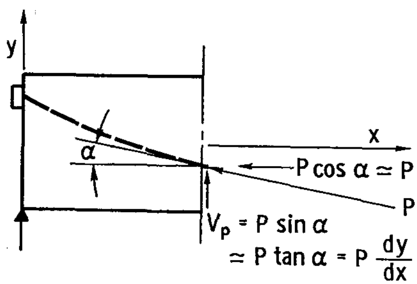


Fig. 1. Vertical component of prestress, V_p, for an inclined tendon.

A design procedure is presented that uses the concept of equivalent loading to obtain the shear component of prestress.
The prestressing tendons are replaced by a set of equivalent loads.
The resulting shear diagram includes the effects of both the vertical component of the prestress and the induced reactions (if any).
Several examples are presented to illustrate the method.
The proposed procedure allows the designer to use familiar analysis techniques, reduce the tedium involved in shear design, and obtain a clear picture of the forces in the concrete.

bility of this approach to shear design is not widely recognized.

Using the procedures presented in this paper, the designer can quickly calculate the shear component of the prestress anywhere in the beam simply by drawing the shear diagram for the equivalent loads. The procedure applies to continuous beams as well as simple spans and automatically includes the effects of induced moments and reactions due to non-concordant tendons.*

Shear by Equivalent Loads

Using the equivalent load concept, the designer looks at the prestressing steel as a cable applying loads to the concrete. The prestress is represented as a system of equivalent vertical loads along the beam length (Figs. 2a and 2b), and as concentrated moments at the supports (Fig. 2c).

The equivalent forces on the beam cause a shear in the concrete. The shear diagram for the equivalent loads represents the shear component of the prestress, V_p.

In this paper a distinction will be made between the *vertical* component of prestress, $P \tan \alpha$, and the *shear* component of prestress which also includes the effect of induced reactions in continuous beams. The distinction is important only for continuous beams with non-concordant tendons.

* A concordant tendon is so located as to produce a line of compressive force in the concrete at each section that coincides with the center of gravity of the steel (cgs). A non-concordant tendon produces a line of compressive force that does not coincide with the cgs.

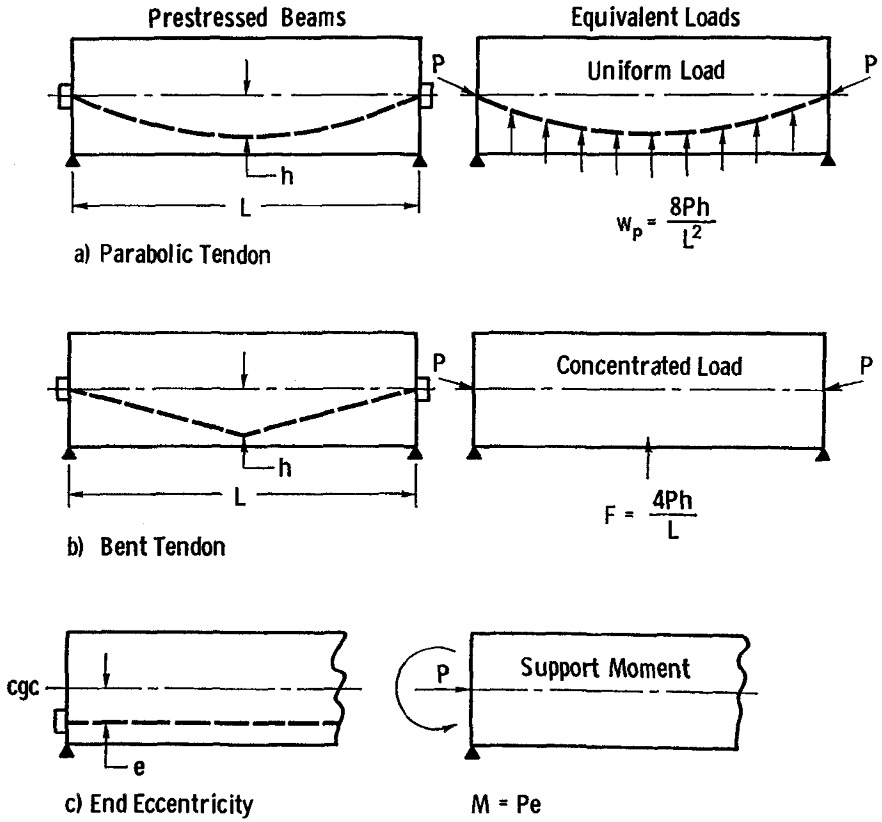


Fig. 2. Equivalent loads due to prestressing.

Losses due to friction will not be included in the discussion that follows, but losses should be included in calculations when they are substantial.

Equivalence to $P\ dy/dx$ for Simple Spans

A short example will help to demonstrate the equivalence of the proposed method with the more common procedure of taking V_p approximately equal to $P \tan \alpha = P\ dy/dx$.

Example 1

The simply supported beam of length L, shown in Fig. 3a, has a parabolic tendon with sag, h, and end eccentricities, e_a and e_b at the left and right supports, respectively.

For an effective prestress, P, the tendon exerts a uniform upward load of $w_p = 8Ph/L^2$ and concentrated end moments of $M_a = Pe_a$ and $M_b = Pe_b$ on the beam as shown in Fig. 3b.

The shear due to the equivalent loads (Fig. 3c) is:

$$V_p = -\frac{1}{2} w_p L + w_p x + \frac{M_b - M_a}{L} \quad (1)$$

For the purpose of comparison, Eq. (1) is rewritten in terms of the prestress force and the tendon geometry:

$$V_p = -\frac{8Ph}{L^2} \bullet \frac{L}{2} + \frac{8Ph}{L^2} x + \frac{Pe_b - Pe_a}{L} \quad (2a)$$

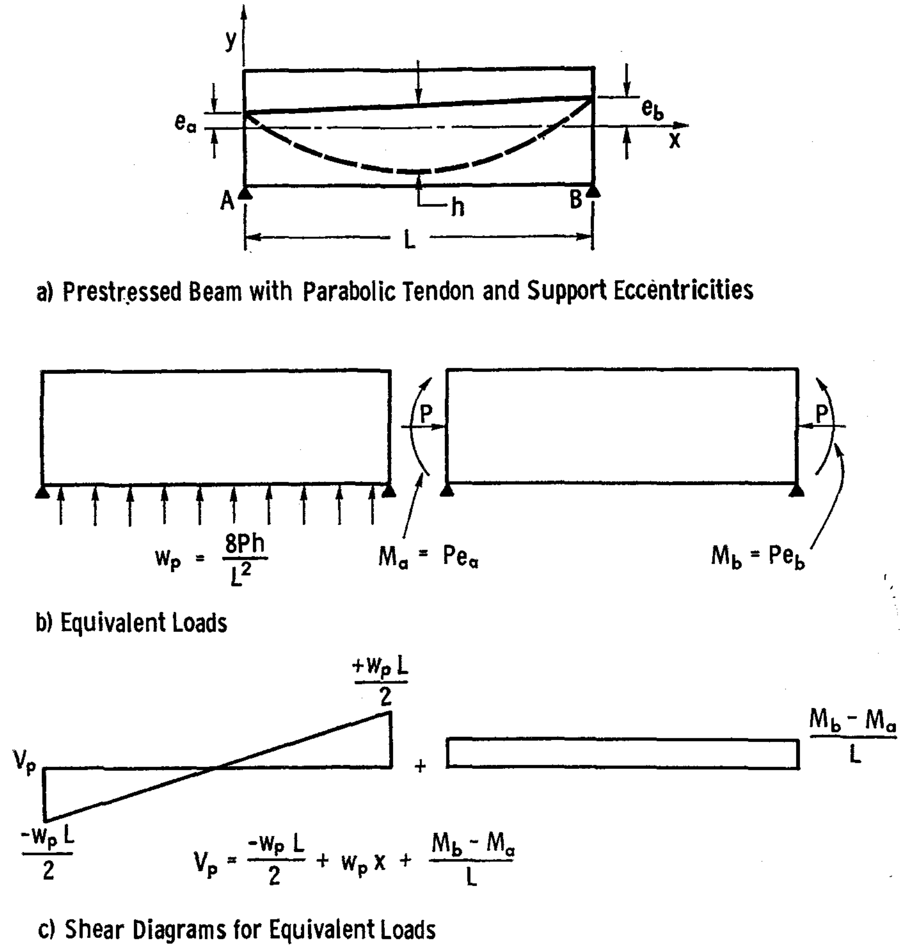


Fig. 3. Diagrams for simply supported beam (Example 1).

from which

$$V_p = P\left(-\frac{4h}{L} + \frac{8hx}{L^2} + \frac{e_b - e_a}{L}\right) \quad (2b)$$

Eq. (2b) may be compared to the value of $V_p = P\,dy/dx$.

The tendon profile in Fig. 3a is described by the equation:

$$y = \frac{4h}{L^2}(x^2 - xL) + \frac{(e_b - e_a)}{L}x + e_a \quad (3)$$

Differentiating and rearranging terms:

$$V_p = P\frac{dy}{dx} = P\left(-\frac{4h}{L} + \frac{8hx}{L^2} + \frac{e_b - e_a}{L}\right) \quad (4)$$

which is identical to the value obtained in Eq. (2b) using the proposed procedure.

The relative advantages of the proposed procedure become distinct when the usable results for the equivalent loading, Eq. (1) and Fig. 3c, are compared with Eq. (4). Fig. 3c is all that most designers need. Fig. 3c and Eq. (1) give a clearer picture of the forces in the concrete than is provided by Eq. (4).

Also, importantly, the proposed procedure allows the designer to use familiar analysis techniques to obtain V_p.

Calculations to obtain the slope of the tendons, which are tedious in comparison, are no longer required.

As attractive as the approach appears for simple spans, it offers added advantages for continuous beams.

Continuous Beams

Two approaches may be taken to obtain the shear in concrete due to prestressing in continuous beams.

Approach 1—The first approach obtains the shear by considering the entire continuous beam under equivalent loading. It may be used for beams with concordant or non-concordant tendons and automatically includes the shear resulting from induced reactions.

Approach 2—The second approach follows the procedure outlined in Example 1 and considers the individual spans as simple spans subjected to equivalent loading. This approach, like the first, may be used for beams with concordant or non-concordant tendons; but for non-concordant tendons, it yields results that *do not* include the effects of induced reactions.

Therefore, for non-concordant tendons it applies only when plastic hinges form and full moment redistribution takes place at ultimate load.

Concordant tendons

The two approaches are equivalent for beams with concordant tendons, because in such beams no support reactions are produced by the prestress. Both approaches produce identical results with those obtained with the more conventional approach, $V_p = P\, dy/dx$.

Example 2 illustrates the two approaches for a continuous beam with concordant tendons.

Example 2

A symmetric two span beam with concordant parabolic tendons is shown in Fig. 4a. For the left span, the cable is described by:

$$y = h\left(-\frac{3x}{L} + \frac{4x^2}{L^2}\right) \qquad (5)$$

The equivalent upward uniform load is:

$$w_p = \frac{8Ph}{L^2} \qquad (6)$$

1. The first approach considers the entire beam subject to an upward load, w_p. The shear diagram for the left span is easily obtained (Fig. 4b) and gives:

$$V_p = -\frac{3}{8} w_p L + w_p x \qquad (7)$$

2. The second approach considers equivalent loads on the individual spans. For the left span, the loads consist of an upward uniform load, w_p, and a counterclockwise support moment at Point B, $M_b = Ph$.

The second approach provides results shown in Fig. 4c and Eq. (8).

$$V_p = -\frac{1}{2} w_p L + w_p x + \frac{Ph}{L} \qquad (8)$$

From Eq. (6):

$$Ph = \frac{w_p L^2}{8} \qquad (9)$$

Eq. (8) then becomes:

$$V_p = -\frac{3}{8} w_p L + w_p x \qquad (10)$$

which is identical to that obtained in Eq. (7) using the first approach.

3. The results in Eqs. (7) and (10) are also equivalent to $P\, dy/dx$:

$$V_p = P\frac{dy}{dx} = Ph\left(-\frac{3}{L} + \frac{8x}{L^2}\right) \qquad (11a)$$

$$= \frac{w_p L^2}{8}\left(-\frac{3}{L} + \frac{8x}{L^2}\right)$$

$$= -\frac{3}{8} w_p L + w_p x \qquad (11b)$$

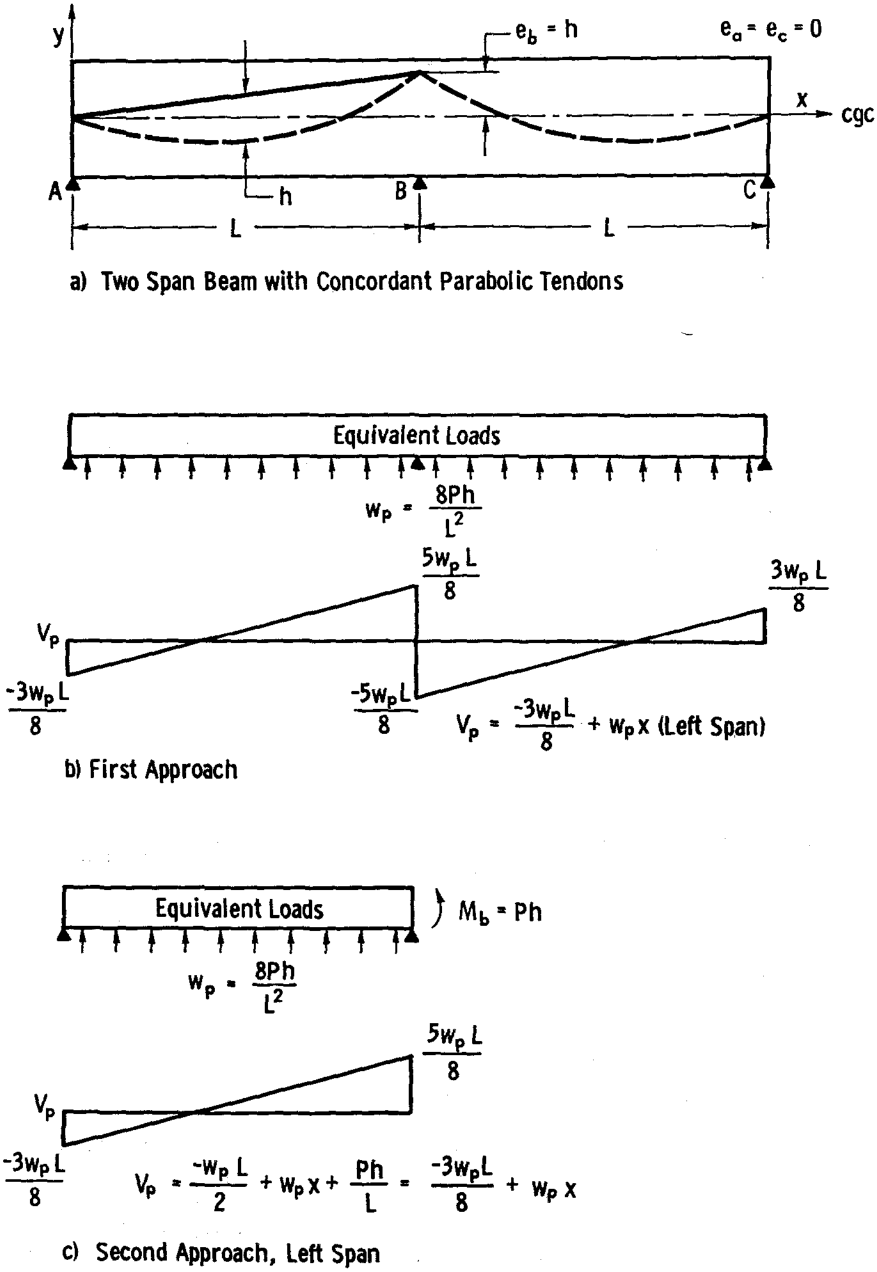


Fig. 4. Diagrams for continuous beam with concordant tendon (Example 2).

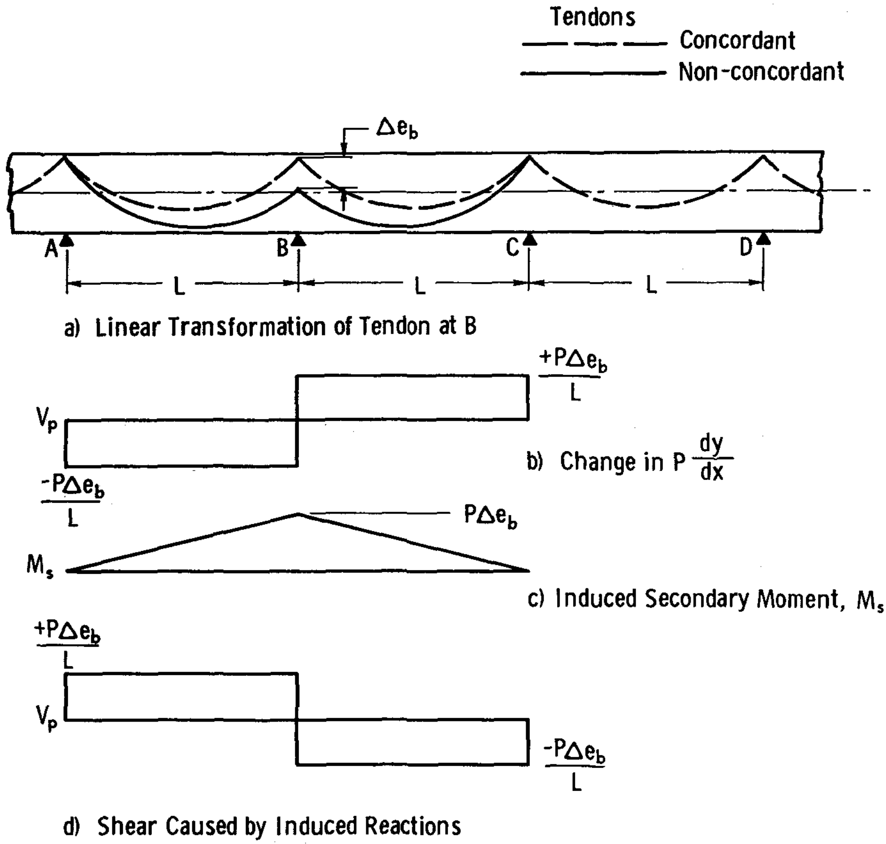


Fig. 5. Effects of linear transformation of concordant tendon on shear component of prestress.

Non-concordant tendons

Non-concordant tendons are of special interest because the prestressing force induces support reactions, shears and moments in the beam. The line of compressive force does not coincide with the center of gravity of steel as it does for concordant tendons.

The induced reactions constitute a real load on the beam, and induced shears and moments must be included in design calculations. The induced reactions are based on an elastic analysis.

If full moment redistribution occurs at ultimate load (unusual for most construction), elastic analysis is no longer applicable. The procedures outlined above for concordant tendons are easily adapted to include induced reactions or to exclude them in the case of full moment redistribution.

For elastic behavior, the shear caused by induced reactions is automatically accounted for by using equivalent loads. This is demonstrated by the continuous prestressed concrete beam shown in Fig. 5a.

In the figure, the concordant tendon in Spans AB and BC is linearly transformed by reducing the support eccentricity at B by an amount Δe_b. The result is a non-concordant tendon. The change in the value of the vertical component of the prestress, $P\ dy/dx$, due to the change in tendon profile equals $-\ P\Delta e_b/L$ and $+\ P\Delta e_b/L$ on Spans AB and BC, respectively (Fig. 5b).

The linear transformation has no effect on the location of the line of compressive force, but it does cause a secondary moment (Fig. 5c) and a secondary shear (Fig. 5d) due to the induced support reactions. The induced shear equals $+ P\Delta e_b/L$ and $- P\Delta e_b/L$ on Spans AB and BC, respectively. The induced shear exactly cancels the change in the vertical component of the prestress.

Therefore, the total shear effect of prestressing remains unchanged and is not affected by the linear transformation. Since the equivalent loads on the continuous beam are not influenced by the linear transformation of the tendon, the shear diagram produced for the equivalent loading will be the same for concordant and non-concordant tendons and will automatically include any induced shear.

If full moment redistribution takes place at ultimate load, the induced reactions disappear at failure. In such a case the shear component of prestress will not include induced shear. V_p may then be calculated either by using the conventional approach or by considering each span individually (Approach 2), as described in Part 2 of the next example.

Although full moment redistribution is rarely used in design, the use of equivalent loads to obtain V_p is just as applicable for plastic design as it is for elastic design.

The amount of moment redistribution actually obtained will vary, depending on the type of construction used. Research by Mattock, Yamazaki, and Kattula[4] indicated that beams with unbonded tendons, and an adequate amount of bonded, non-prestressed reinforcement may exhibit somewhat more inelastic moment redistribution than similar beams with bonded tendons.

In either case, when the amount of moment redistribution is limited to that allowed by the ACI Code,[1] the designer should use Approach 1 for calculating V_p.

At ultimate load, a properly proportioned beam will fail in flexure, not shear. Therefore, the value of prestress used to calculate the equivalent loads for shear design should, logically, reflect the tendon force at the ultimate flexural load. In beams with unbonded tendons, the prestress force will increase as the ultimate bending moment is attained at the critical sections.

The increase in prestress is almost uniform throughout the beam due to tendon slip and may be estimated using ACI Code Eq. (18-4):

$$f_{ps} = f_{se} + 10{,}000 + \frac{f'_c}{100\rho_p}$$

where

f_{ps} = stress in prestressing steel at design load

f_{se} = effective stress in prestressing steel

f'_c = strength of concrete

ρ_p = ratio of prestressed reinforcement

This increased stress may be included in the calculation of the equivalent loads and V_p. In most beams, however, the increase may be conservatively neglected. For bonded tendons, the increase in prestress is quite localized and should be neglected for shear design.

Example 3 illustrates the use of equivalent loads to obtain the shear component of prestress for a beam with non-concordant tendons. It demonstrates the simplicity with which the proposed procedures may be applied to shear design.

Example 3

A symmetric two-span continuous T-beam with bonded tendons is shown in Fig. 6a.[5] The beam has non-con-

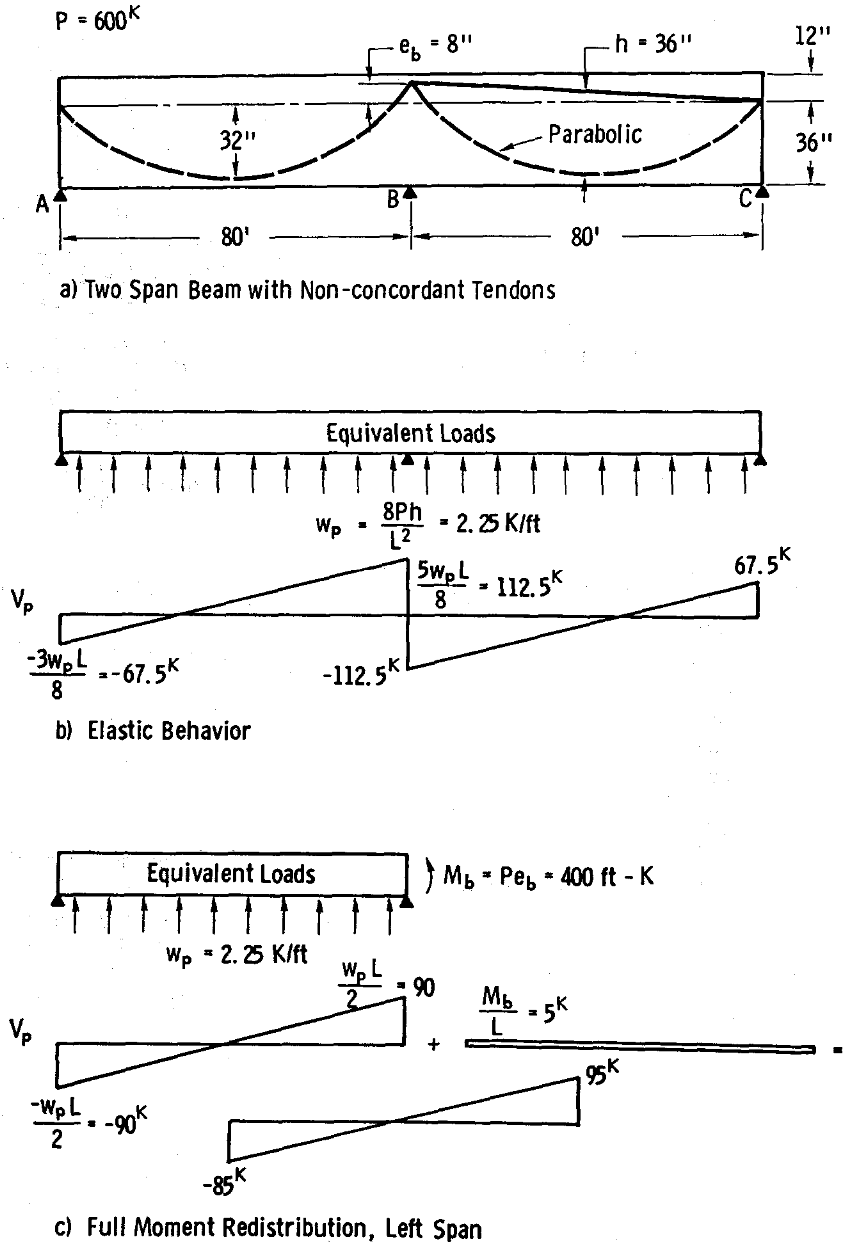


Fig. 6. Diagrams for continuous beam with non-concordant tendon (Example 3).

cordant parabolic tendons with sag, $h = 3$ ft, and prestress force, $P = 600$ kips.

The total upward load is:

$$w_p = \frac{8Ph}{L^2} = \frac{8 \times 600 \times 3}{80^2}$$

$$= 2.25 \text{ kips per ft}$$

1. The shear component of the prestress, V_p, is obtained for elastic behavior (i.e., less than full moment redistribution) by calculating the shear in the beam due to the load w_p. The solution (Fig. 6b) shows that V_p varies linearly from $-3w_pL/8 = -67.5$ kips at the left support to $+5w_pL/8 = 112.5$ kips at the center support. As calculated, V_p includes the effects of the induced support reactions.

2. Should the beam be designed for full moment redistribution at ultimate load, V_p may be calculated using equivalent loads on the *individual spans*. As shown in Fig. 6c, each span is considered as a *simply supported span* for purposes of the calculations. Since the tendon is bonded, the increase in P, near ultimate can be neglected.

Looking at the left span, the equivalent loads consist of a uniform load, $w_p = 2.25$ kips per ft, and a support moment:

$$M_b = Pe_b = 600 \times 8/12 = 400 \text{ ft-kips}$$

The shear diagrams produced by these loads are also shown in Fig. 6c. The final results show that V_p varies linearly from $-w_pL/2 + M_b/L = -85$ kips to $+w_pL/2 + M_b/L = 95$ kips. These values are identical with those obtained from $V_p = P\,dy/dx$ and do not include induced support reactions.

The differences in Solutions 1 and 2 are due to the induced support reactions. Since most beams are proportioned for flexure based on elastic behavior, Solution 1 represents the type of solution that will be commonly used in practice. Solution 1 should, also, be used if partial moment redistribution is planned, as allowed by the ACI Code.[1]

If the beam in the example utilizes unbonded, rather than bonded tendons, the analysis procedures will be identical to those presented above, with the exception that the effective prestress, P, will increase as the ultimate load is attained.

Assuming that $f_{se} = 150{,}000$ psi, $f'_c = 5000$ psi and $\rho_p = 0.01$, and using Eq. (18-4) of the ACI Code:[1]

$$f_{ps} = f_{se} + 10{,}000 + \frac{f'_c}{100\rho_p}$$

$$= 150{,}000 + 10{,}000 + \frac{5000}{100 \times 0.01}$$

$$= 165{,}000 \text{ psi}$$

The value of P at ultimate is, therefore $P = (165/150)600 = 660$ kips.

This results in a 10 percent increase in the equivalent loads and in the shear component of prestress, V_p.

Reversed curvature

The beams presented in the preceding sections have harped tendons at the supports for illustrative purposes. In practice, the sharp angle change is replaced by a portion of the tendon in which the direction is gradually changed over the supports (Figs. 7 and 8).

The portion of tendon with the reversed curvature exerts a downward distributed load on the beam. The effect of the reversed curvature must be taken into account in shear design.

Equivalent loads are illustrated for parabolic tendons with reversed curvature in Figs. 7 and 8.[6,7] For an interior span, the uniform upward load is:

$$w_p = \frac{8Pc}{(1 - 2a)L_i^2} \tag{12}$$

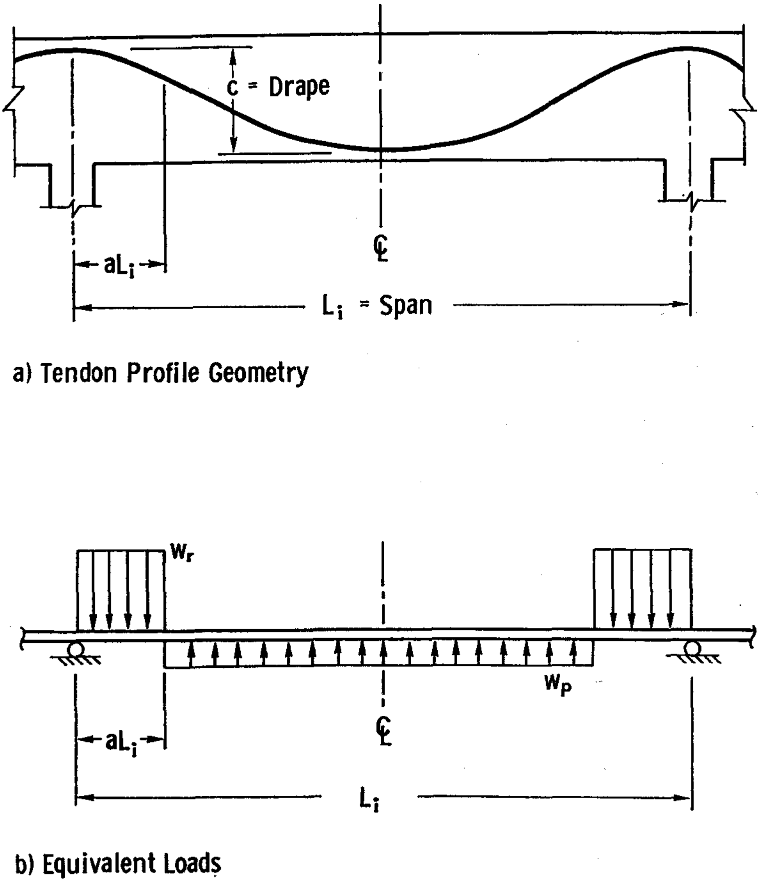


Fig. 7. Typical interior span.

where

c = drape of tendon profile, high point to low point

a = ratio of reverse curve length span length L_i

The uniform downward load for the reversed segment is:

$$w_r = \frac{1-2a}{2a} w_p \tag{13}$$

An exterior span is likely to have a change in curvature near the midspan which results in different upward loads on either side of the low point in the tendon profile. For the exterior span shown in Fig. 8, the upward uniform load due to the external parabolic segment is:

$$w_{pe} = \frac{2Pd}{b^2 L^2_e} \tag{14}$$

where

d = drape of tendon profile at end segment of exterior span

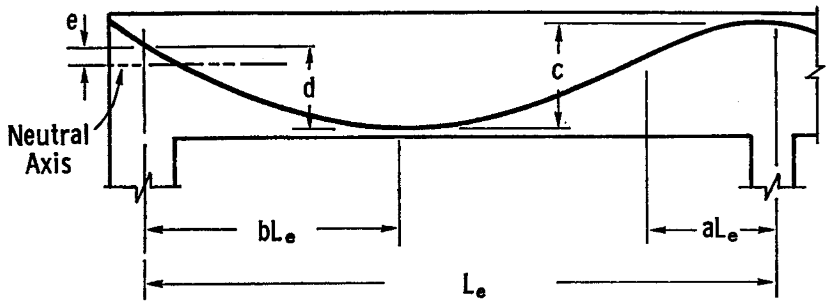
b = ratio of end segment length to span length in exterior span, L_e

The balance of the uniform upward load is:

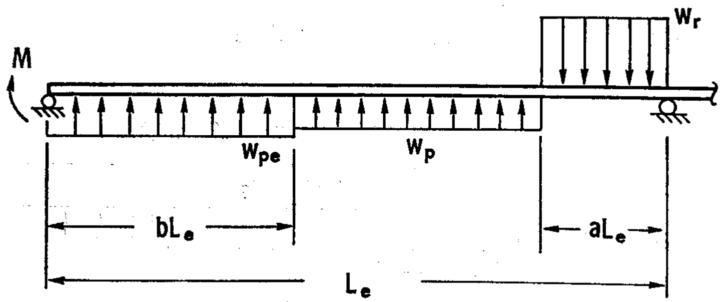
$$w_p = \frac{2Pc}{(1-b)(1-b-a)L^2_e} \tag{15}$$

The downward load for the reversed segment is given by Eq. (13).

These equivalent loads are used to calculate the shear component of the



a) Tendon Profile Geometry



b) Equivalent Loads

Fig. 8. Typical exterior span.

prestress, as is demonstrated in Example 4.

Example 4

The symmetric two-span beam in Example 3 is redesigned with a reverse curve in the tendon over the center support as shown in Fig. 9a. For the external segment, with tendon drape, $d = 32$ in., and prestress force, $P = 600$ kips, the upward uniform load is:

$$w_{pe} = \frac{2Pd}{b^2L^2_e} = \frac{2 \times 600 \times 32/12}{(0.50)^2 \times 80^2}$$

$$= 2.0 \text{ kips per ft}$$

For the inner segment, with tendon drape, $c = 40$ in., upward load is:

$$w_p = \frac{2Pc}{(1-b)(1-b-a)L^2_e}$$

$$= \frac{2 \times 600 \times 40/12}{(0.50)(0.40) \times 80^2}$$

$$= 3.125 \text{ kips per ft}$$

The downward load for the reversed segment is:

$$w_r = \frac{1-2a}{2a}\, w_p$$

$$= \frac{1 - 2 \times 0.10}{2 \times 0.10} \times 3.125$$

$$= 12.5 \text{ kips per ft}$$

As before, V_p can be obtained by

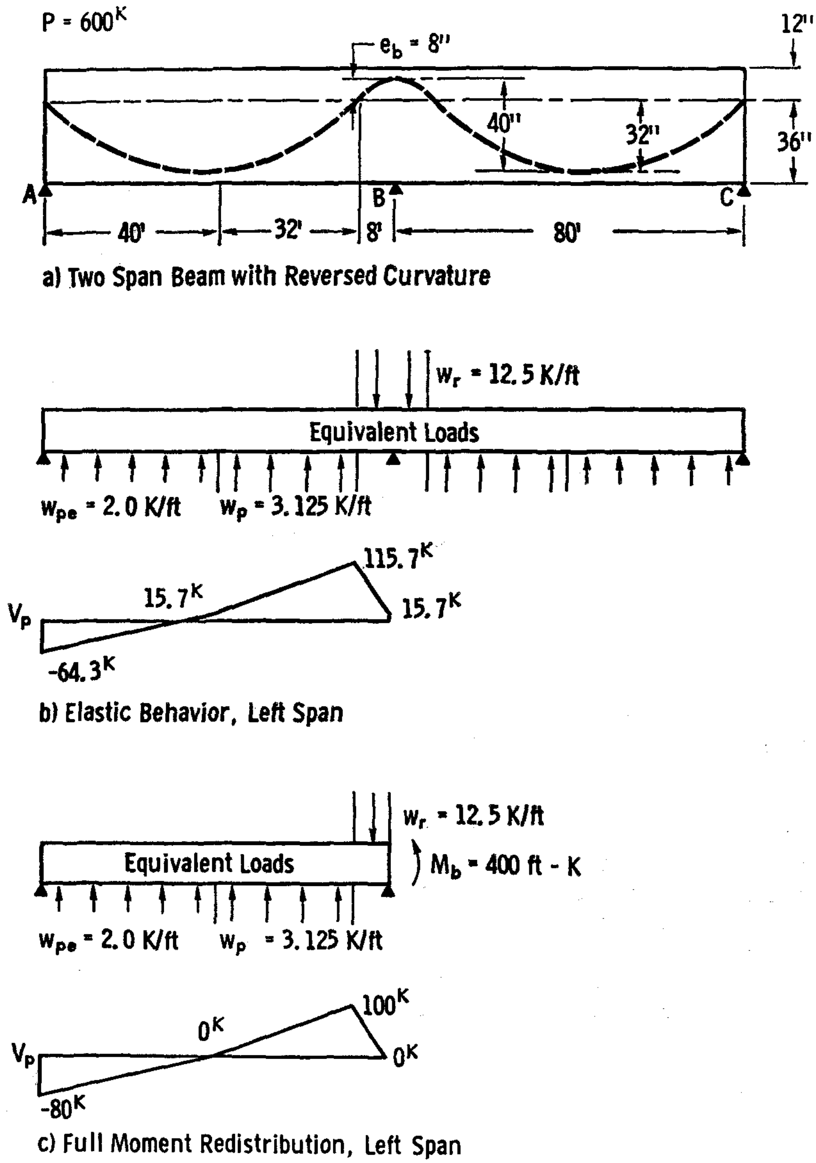


Fig. 9. Diagrams for continuous beam with reversed curvature (Example 4).

constructing the shear diagram for the beam subjected to the equivalent loads.

The calculation of V_p for beams with reversed curvature can be greatly simplified by using the moment coefficients provided in References 6 and 7. The solution, shown for the left span (Fig. 9b), includes the shear due to the induced reactions.

The effect of the change in the tendon profile on V_p may be seen by comparing Figs. 6b and 9b. The shear diagram for full moment distribution (Fig. 9c) can be obtained by loading each span individually, using Approach 2.

Conclusions

1. In a prestressed concrete beam, V_p, the shear component of the prestress due to both the vertical component of prestress and the induced reactions (if any), may be obtained by replacing the tendon with a set of equivalent loads on the beam and calculating the shears so produced.

2. While the examples presented in this paper are limited to parabolic tendons and two-span continuous beams, the methods illustrated may be applied to any tendon profile.

3. The use of equivalent loads to calculate the shear component of the prestress, V_p, has several advantages over the conventional method:

a. The proposed procedures allow the designer to use familiar analysis techniques to obtain V_p and thereby reduce the tedium involved in shear design.

b. The use of equivalent loads gives a clearer picture of the forces in the concrete.

c. The proposed method (Approach 1) automatically includes the effects of induced reactions in continuous beams with non-concordant tendons. The method is also, easily adapted (Approach 2) for full moment redistribution.

References

1. ACI Committee 318, "Building Code Requirements for Reinforced Concrete (ACI 318-71)," American Concrete Institute, Detroit, Michigan, 1971, 78 pp.
2. Moorman, R. B. B., "Equivalent Load Method for Analyzing Prestressed Concrete Structures," *ACI Journal*, V. 23, No. 5, January 1952, pp. 405-416.
3. Lin, T. Y., "Load Balancing Method for Design and Analysis of Prestressed Concrete Structures," *ACI Journal*, V. 60, No. 6, June 1963, pp. 719-742.
4. Mattock, A. H., Yamazaki, J., and Kattula, B. T., "Comparative Study of Prestressed Concrete Beams, With and Without Bond," *ACI Journal*, V. 68, No. 2, February 1971, pp. 116-125.
5. Lin, T. Y., and Thornton, K., "Secondary Moment and Moment Redistribution in Continuous Prestressed Concrete Beams," PCI JOURNAL, V. 17, No. 1, January-February 1972, pp. 8-20.
6. *Post-Tensioning Manual*, Post-Tensioning Institute, Glenview, Illinois, 1976, 288 pp.
7. Turula, P., and Freyermuth, C. L., "Moment Influence Coefficients for Continuous Post-Tensioned Structures," PCI JOURNAL, V. 17, No. 1, January-February 1972, pp. 35-57.

Discussion of this paper is invited. Please forward your comments to PCI Headquarters by Sept. 1, 1977.