# OPEN FORUM
## PROBLEMS AND SOLUTIONS

## Continuous Prestressed Concrete Beams

**Q1:** *What is the difference in behavior between a simple span prestressed concrete beam and a continuous prestressed concrete beam, under the action of the prestress force?*

**A1:** When a simple span beam is prestressed eccentrically, it is free to deform. In general, there are transverse deflections, the beam ends rotate and the beam shortens. No external reactions are set up by prestressing. The prestressing moment at any section is simply:

$$M_p = Pe$$

where $P$ is the prestressing force and $e$ is the eccentricity of the resultant prestress force, measured from the centroid of the section (the c.g.c.).

When a continuous beam is prestressed, it is not free to deform under the action of the prestressing force. The deformations are restrained at the supports and external reactions are set up. These reactions produce bending moments in the beam, often called "secondary moments." The resultant moment at any section of the beam due to prestressing is then given by:

$$M_p = Pe + M_s$$

where $M_s$ is the secondary bending moment.

For example, consider a continuous beam with two equal spans, prestressed by a group of tendons having constant eccentricity. If the beam were not attached to the center support (or held in contact with it by gravity), it would deflect upwards an amount $\delta$:

$$\delta = \frac{Pe(2L)^2}{8EI}$$

If, however, the beam is restrained in contact with the center support, a downward restraint force, $R$, sufficient to cause a deflection $-\delta$ must be acting at the center support:

$$-\delta = \frac{R(2L)^3}{48EI}$$

Equating the two deflections:

$$\frac{Pe(2L)^2}{8E} = \frac{R(2L)^3}{48EI}$$

the reaction, $R$, is found to equal $R = 3Pe/L$.

For equilibrium, this downward force is counterbalanced by equal upward forces $R/2$ acting at each of the end supports. These support reactions, and the secondary bending moments caused in the beam by them, will be as shown in Figs. 1a and 1b. The resultant moments caused by prestressing ($Pe + M_s$) will be as shown in Fig. 1c.

Despite the term "secondary," it should be noted that the moments and stresses arising from this restraint can be considerable. Because the algebraic sum of the secondary reactions is zero, for this case, the secondary moment at the interior support is 150 percent of the primary moment and of opposite sign. It is important to note that because the secondary moment is caused by the induced reactions, the secondary moment always varies linearly between the supports.

The secondary bending moment causes a displacement of the effective point of application of the prestress force $P$ by an amount varying linearly from zero at the outer supports, to a maximum of $v = (M_s/P)$ at the middle support, as shown in Fig. 1d. A similar displacement of the effective line of pressure occurs in all structures where secondary bending moments are caused by prestressing. However, it is possible to design a tendon profile such that no secondary bending moments are set up, and hence the line of pressure and the c.g.s. line coincide. Such a profile is called a concordant profile, after Guyon[1] who first used the term.

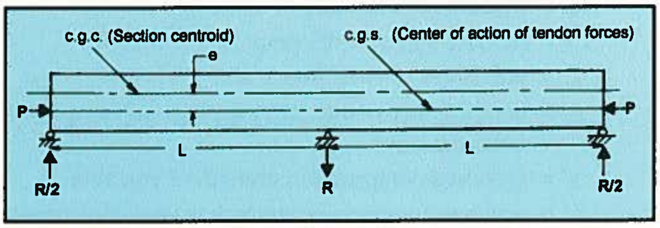


Fig. 1a. Restraint forces due to prestress.

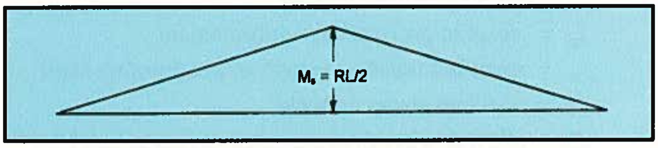


Fig. 1b. Secondary bending moments caused by restraint forces.

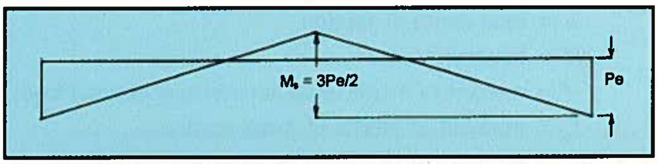


Fig. 1c. Resultant moments due to prestressing.

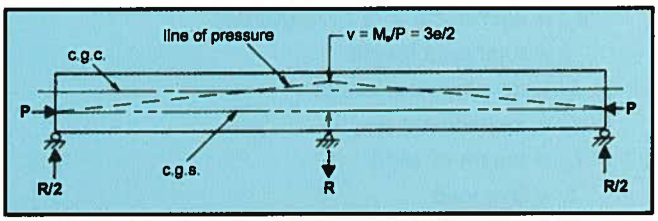


Fig. 1d. Displacement of line of pressure due to secondary moments.

The magnitude of the prestressing moments $M_p$ can be calculated by analyzing the beam under the action of the resultant lateral forces produced by the prestressing tendons, using any method of structural analysis. The secondary moments can be obtained directly if the flexibility method of analysis is used, making the beam statically determinate by rotational releases at the supports, as in Example A. This approach lends itself to the design of a c.g.s. line that results in concrete stresses less than the specified limiting stresses all along the length of the beam. Also, using this approach, the c.g.s. line can be any convenient arbitrary shape and variation of the beam cross section and of the prestress force along the beam can readily be taken into account.

Note that all prestressed concrete beams must be free to shorten axially due to elastic and creep shortening of the concrete under the action of the prestress force, otherwise some of the prestress force will not get into the beam but will cause axial tension restraint forces at the ends of the beam. Customary design calculations implicitly assume this freedom to shorten axially.

## EXAMPLE A

Calculate the secondary moments and resultant prestressing moments for the tendon profile shown in Fig. A1. The tendon force is 250 kips (1112 kN).

Make the beam statically determinate by inserting a moment release (i.e., a hinge) at C. Let a moment $X_1 = 1$ act at C. The distribution of moments due to $X_1 = 1$ will be as shown in Fig. A2.

The $(M_0)_p$ diagram of the primary prestressing moments $Pe$ may conveniently be broken up into the components shown in Fig. A3, for purposes of calculating the displacement $u_p$ at the hinge due to the prestressing moments $Pe$.

The displacement $f_{11}$ at the hinge when moment $X_1 = 1$ is given by:

$$f_{11} = \int \frac{M_1^2}{EI} ds = \frac{2}{EI}\left\{\left[\frac{1(100)}{2}\right]\left(\frac{2}{3}\right)\right\} = +66.7/EI$$

$$u_p = \int \frac{M_1 (M_0)_p}{EI} ds = \frac{1}{EI}\left\{\left[\frac{-1.44P(100)}{2}\right]\left(\frac{53.3}{100}\right)(1) + \left[\frac{2}{3}(-1.10P)(100)\right]\frac{1}{2}\right\}$$

$$+ \frac{2}{EI}\left\{\left[\frac{0.4P(100)}{2}\right]\left(\frac{2}{3}\right)\right\} + \frac{1}{EI}\left\{\left[\frac{0.5P(100)}{2}\right]\left(\frac{1}{3}\right)\right\}$$

$$= \frac{P}{EI}(-38.4 - 36.7 + 26.7 + 8.3) = -40.1\,P/EI$$

Now if $X_1$ is the restoring moment required for continuity at C:

$$X_1 f_{11} + u_p = 0$$

Therefore:

$$X_1 = -\frac{u_p}{f_{11}} = -\left(\frac{-40.1P/EI}{+66.7/EI}\right)$$

$$= +0.60P \text{ kip-ft } (0.81 \text{ kN-m})$$

At C:

Secondary moment = $X_1$ = +0.60$P$ kip-ft = +0.60(250) = +150 kip-ft (203 kN-m)

Shift in center of pressure at C:
$v_c$ = +0.60 ft (0.18 m)

Therefore, the ordinate of line of pressure at C:
$(e_c + v_c)$ = (+0.60 + 0.40) = +1.00 ft (0.30 m) $(e_c')$

The prestressing moment at C:
$(M_p)_c = Pe_c'$ = 250(1.00) = 250 kip-ft (339 kN-m)

The secondary moment at other locations along the beam can be found by proportion. The location of the line of pressure and the magnitude of the prestressing moments at other points can be calculated in the same manner as at Point C. The line of pressure is shown in Fig. A4.

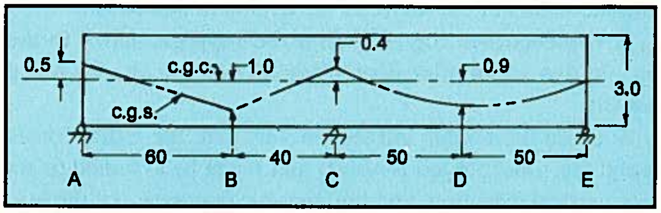


Fig. A1. Tendon profile.

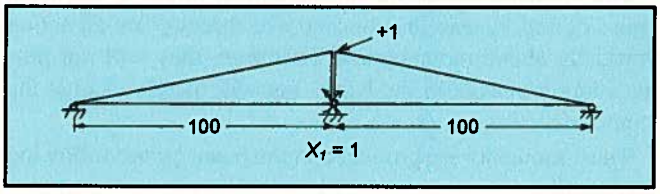


Fig. A2. $M_1$ diagram.

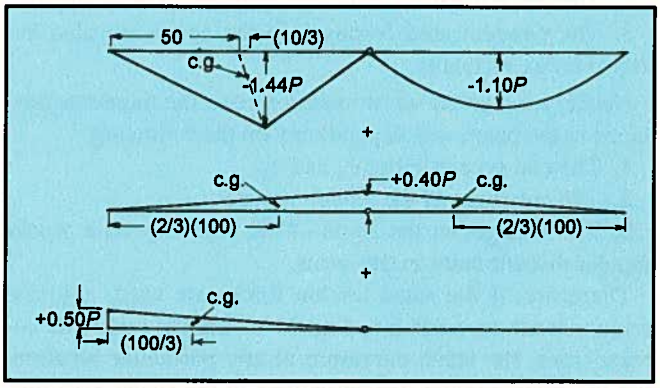


Fig. A3. Components of prestressing moment $(M_0)_p$.

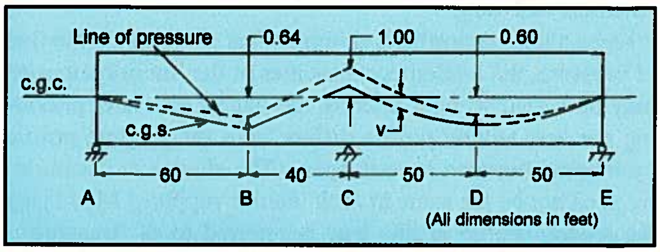


Fig. A4. Displacement of line of pressure from the c.g.s. line.

**Q2:** *What is meant by "transformation" of the prestressing tendon profile (the c.g.s. line) in a continuous prestressed concrete beam? Why is transformation useful?*

**A2:** It is necessary to further discuss the effect of the tendon forces on a continuous prestressed concrete beam. The following assumptions are made:

**1.** The eccentricities of the prestressing tendons are small compared with the length of the member.

**2.** At any particular location, the prestress force is constant (it may vary along the length of the member).

**3.** The action of the tendon on the concrete will be as shown in Fig. 2 (vertical scale much exaggerated.)

Forces acting on the concrete are:

**1.** Forces $P$ acting at each end of the beam, in a direction tangential to the line of the tendon (the c.g.s. line).

**2.** Radial forces $q$ normal to the tendon, at any point = $P/r$ where $P$ and $r$ are, respectively, the resultant prestress force and the radius of curvature of the tendon at that point.

**3.** Concentrated forces $Q$ over the supports and $F$ in the spans, due to angular discontinuities in the shape of the tendon.

Because the tendon curves are very flat, the radial forces $q$ and the concentrated forces $Q$ and $F$ can be assumed to act in a vertical direction, and horizontal components of the tendon forces at the beam ends $P'A$ and $P'B$ can be taken equal to the tendon forces $P_A$ and $P_B$.

The vertical components of the tendon forces at the beam ends, $V_A$ and $V_B$, and the concentrated forces $Q$ are all acting vertically above the supports. Therefore, they will not produce any moments in the beam, but will merely change the support reactions.

Thus, moments are produced in the beam by the following forces only:

**1.** Tendon forces $P_A$ and $P_B$ acting with eccentricities $e_A$ and $e_B$ at A and B, respectively.

**2.** The radial forces $q$, which are assumed to act vertically.

**3.** The concentrated forces $F$ in the spans are also assumed to act vertically.

Hence, for a given set of tendon forces, the moments produced in the beam will depend only on the following:

**1.** The end eccentricities $e_A$ and $e_B$.

**2.** The curvature of the tendon profile.

**3.** The changes in the slope of the tendon profile at any angular discontinuity in the spans.

Therefore, if the same tendon forces are used, any two tendons (such as a and b in Fig. 3) having the same end eccentricities, the same curvature at any particular location, and the same angular discontinuities in the spans, will produce the same prestressing moments in the beam and will, therefore, have the same line of pressure. Only the support reactions will vary.

Once a tendon profile is designed that gives a suitable line of pressure, the tendon eccentricities at the interior supports may be varied without altering the line of pressure, providing the new tendon profile differs from the original profile by a linear function in each span. (The change of eccentricity need not be the same at each interior support.) Modifying the tendon profile in this way is referred to as "transforming" the profile. When a tendon profile is transformed in this way, its curvature at any point will be unchanged because the curves are very flat and the curvature can be taken as $d^2y/dx^2$. Similarly, the angular discontinuities will be unchanged because the angles involved are small.

Transformation of tendon profiles is often very useful in arriving at convenient final locations for the tendons, while still maintaining the line of pressure in a location that results in the stresses in the member not exceeding allowable values under all loading conditions. For instance, it may be found that to satisfy stress limits, the line of pressure must lie above the top of the beam in the vicinity of interior supports. The actual c.g.s. line can be brought within the beam by transforming the line of pressure downwards by whatever amount is necessary to provide adequate cover to the tendons.

An example of this is seen in Fig. 4, which shows (with much exaggerated vertical scale) one-half of a continuous T-beam having three spans of 60 ft (18.3 m) each. The "Tendon Zone" defines the region within which the line of pressure must lie if the specified concrete stress limits are not to be exceeded under all load conditions. The boundaries of the tendon zone are a function of the section properties, the initial and final prestress forces, and the maximum and minimum moments acting under service load conditions. The final c.g.s. line is a downward transformation of

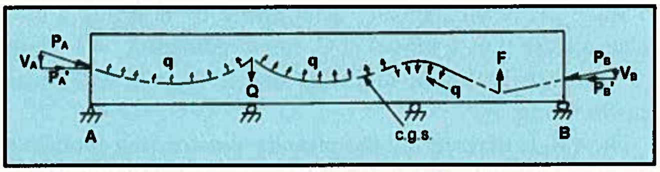


Fig. 2. Forces from the tendon acting on the concrete.

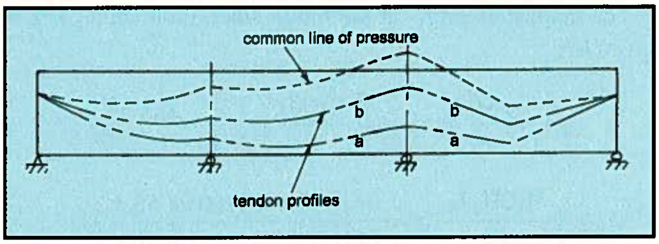


Fig. 3. Common line of pressure for linearly transformed tendon profiles.

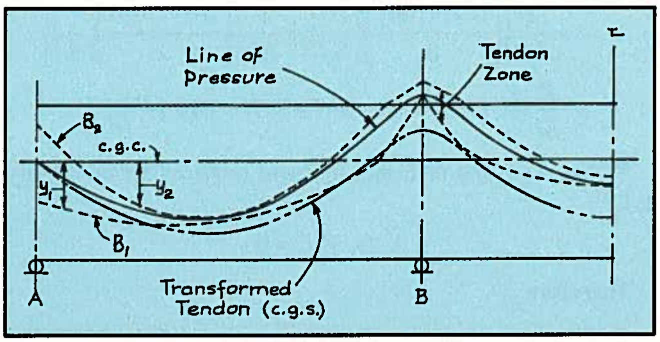


Fig. 4. Example of a continuous beam in which c.g.s. line is transformed downward from the line of pressure in order to provide sufficient cover over the tendon.

the line of pressure, which everywhere lies inside the tendon zone.

Downward transformation of the tendon profile can be used to reduce the reactions on the interior support bents of a continuous prestressed concrete bridge. It can also reduce the angular discontinuity of the c.g.s. line at interior supports and so reduce the loss of prestress in interior spans due to friction at the time of prestressing.

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**Q3:** *The "Tendon Zone" is the region within which the line of pressure must lie if the specified limiting concrete stresses under service load are not to be exceeded under all loading conditions. How are the boundaries of the "Tendon Zone" determined?*

**A3:** The effect of applying a moment $M$ to a prestressed beam is to produce a resulting stress distribution identical to that which would occur if the point of application of the prestressing force $P$ were moved a distance $M/P$.

Let $M_1$ be the least (i.e., the minimum positive or maximum negative) and $M_2$ be the greatest (i.e., the maximum positive or minimum negative) value algebraically of the moments to which a section is subject. The center of thrust will, therefore, occupy two extreme positions $E_1$ and $E_2$ corresponding to displacements $M_1/P$ and $M_2/P$, respectively, as shown in Fig. 5.

For given stress limits at the top and bottom edges of the beam section, limiting values for the position of $E_1$ and $E_2$ can be calculated. (For example, for a rectangular section beam in which no tension is to occur, the limiting positions for $E_1$ and $E_2$ are the middle third points of the section, i.e., the kern points.) Let the upper and lower limits be C and C′, respectively, and their distances from the centroid of the section be $c$ and $c'$, as shown in Fig. 5. The region between C and C′ is referred to as the "limiting zone."

On the cross section (shown in Fig. 6), mark off distances $-M_2/P$ and $-M_1/P$ from C and C′, respectively, giving Points $B_2$ and $B_1$. The center of pressure of the prestress acting alone ($E_o$) must lie between $B_1$ and $B_2$; if $E_o$ is below $B_1$ then under moment $M_1$, the center of thrust would lie below C′, and if $E_o$ is above $B_2$ then under moment $M_2$, the center of thrust would lie above C. In both of these cases, the permissible stresses would be exceeded.

The distance from the centroid of the section to $B_1$ is $y_1 = (-M_1/P - c')$.

The distance from the centroid of the section to $B_2$ is $y_2 = (-M_2/P + c)$.

If we mark the lines $B_1$ and $B_2$ at distances $y_1$ and $y_2$ from the centroid of the section (see Fig. 4) on the elevation of the beam, the line of pressure due to prestress only must lie between these lines if the allowable stresses are not to be exceeded. The region between Lines $B_1$ and $B_2$ is referred to as the tendon zone. In a statically determinate beam, the c.g.s. line must always lie in the tendon zone. The concept of the tendon zone was originated by Guyon.[1]

The distances $c$ and $c'$ may be calculated in terms of the limiting stresses at either the top or the bottom of the section, whichever is convenient.

If the resultant force acts at the upper boundary of the limiting zone (Fig. 7a):

$$f_a = f_g\left(1 + \frac{cy_a}{r_2}\right) \quad \text{and} \quad f_b = f_g\left(1 - \frac{cy_b}{r^2}\right)$$

Therefore:

$$c = \frac{r^2}{y_a}\left(\frac{f_a}{f_g} - 1\right) \quad \text{or} \quad c = \frac{r^2}{y_b}\left(1 - \frac{f_b}{f_g}\right)$$

If the resultant force acts at the lower boundary of the limiting zone (Fig. 7b):

$$f_a = f_g\left(1 - \frac{c'y_a}{r_2}\right) \quad \text{and} \quad f_b = f_g\left(1 + \frac{c'y_b}{r^2}\right)$$

Therefore:

$$c' = \frac{r^2}{y_a}\left(1 - \frac{f_a}{f_g}\right) \quad \text{or} \quad c' = \frac{r^2}{y_b}\left(\frac{f_b}{f_g} - 1\right)$$

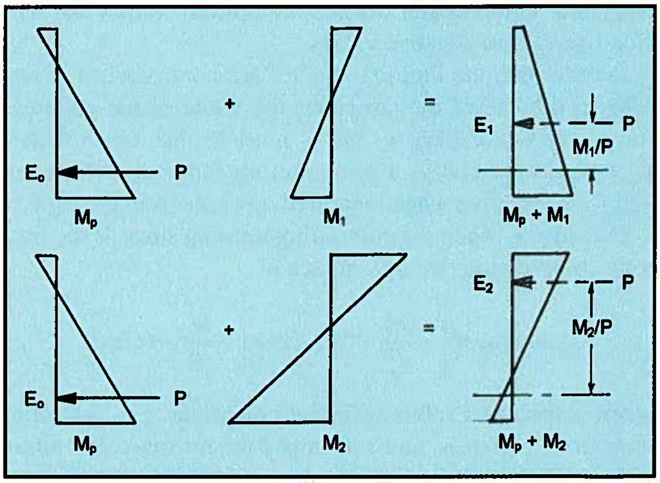


Fig. 5. Change in stress distribution due to applied moments.

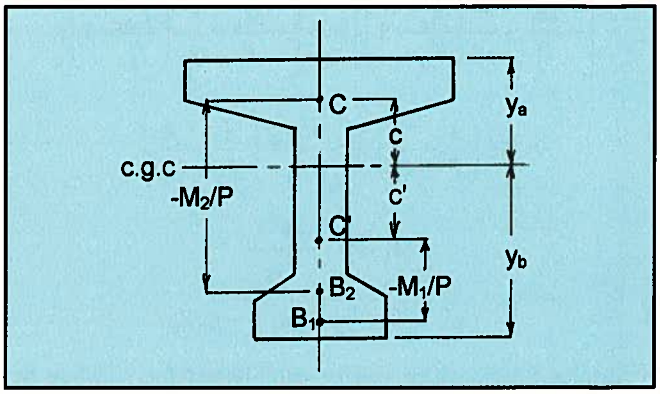


Fig. 6. Typical section.

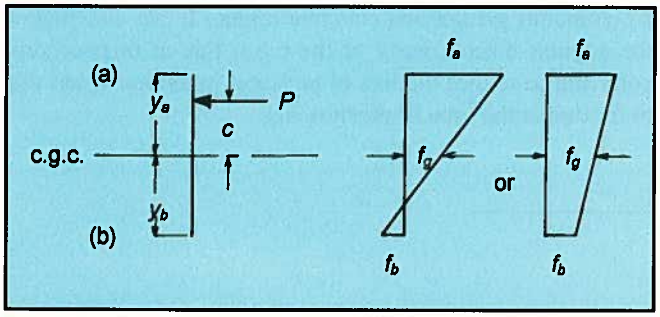


Fig. 7a. Possible stress distributions when the resultant force acts at the upper boundary of the limiting zone.

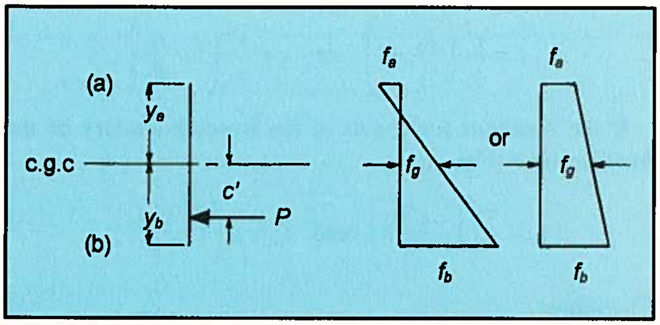


Fig. 7b. Possible stress distributions when the resultant force acts at the lower boundary of the limiting zone.

Table 1. Relationships for determining value of prestress force to be used when calculating $y_1$ and $y_2$.

| | |
|---|---|
| $y_1$ | (a) If the controlling limiting stress is always (i.e., initially and finally) the compression stress at $f_{cb}$ at face b, use: $P_i$ if $M_1$ is $> -f_{cb}Z_b$, otherwise use $P$<br>(b) If the controlling limiting stress is always the tension (or minimum compression) stress $f_{ta}$ at face a, use: $P_i$ if $M_1$ is $> +f_{ta}Z_a$, otherwise use $P$<br>(c) If the controlling limiting stress is $f_{cb}$ initially and $f_{ta}$ finally, $y_1$ must be calculated using both initial and final values of the prestress force, and the algebraically larger value of $y_1$ is to be used. |
| $y_2$ | (a) If the controlling limiting stress is always the compressive stress $f_{ca}$ at face a, use: $P_i$ if $M_2$ is $< +f_{ca}Z_a$, otherwise use $P$<br>(b) If the controlling limiting stress is always the tension (or minimum compression) stress $f_{tb}$ at face b, use: $P_i$ if $M_2$ is $< -f_{tb}Z_b$, otherwise use $P$<br>(c) If the controlling limiting stress is $f_{ca}$ initially and $f_{tb}$ finally, $y_2$ must be calculated using both the initial and final values of the prestress force, and the algebraically smaller value of $y_2$ is to be used. |

**Note:** When substituting numerical values in the above relationships, compression limiting stresses are entered as positive values and tension limiting stresses are entered as negative values, e.g., if $f_{tb}$ = 200 psi (1379 kPa) tension, $-f_{tb}Z_b = -(-200)Z_b = 200Z_b$, a positive quantity.

Note that when substituting numerical values in the above equations, compression stresses are positive values and tension stresses are negative values.

In order that the limiting stresses are not exceeded at any time in the life of the structure, the value of the prestress force used to calculate $y_1$ and $y_2$ must be that which makes $y_1$ a maximum and $y_2$ a minimum algebraically. (Note: $y_1$ and $y_2$ are positive when measured upwards from the c.g.c.)

Consider $y_1$ when the controlling limiting stress is the limiting compression stress $f_{cb}$ at face b:

$$y_{1i} \text{ is} > y_{1f} \text{ if} \left[-\frac{M_1}{P_i} - (c')_i\right] \text{is} > \left[-\frac{M_1}{P} - (c')_f\right]$$

where subscript $i$ refers to initial conditions, i.e., just after transfer of prestress, and subscript $f$ (or no subscript) refers to final conditions, i.e., after all losses of prestress.

Then $y_{1i}$ is $> y_{1f}$ if:

$$\left[-\frac{M_1}{P_i} - \frac{r^2}{y_b}\left(\frac{f_{cb}}{f_{gi}} - 1\right)\right] \text{is} > \left[-\frac{M_1}{P} - \frac{r^2}{y_b}\left(\frac{f_{cb}}{f_g} - 1\right)\right]$$

$$M_1\left(\frac{1}{P} - \frac{1}{P_i}\right) \text{is} > \frac{r^2 f_{cb}}{y_b}\left(\frac{1}{f_{gi}} - \frac{1}{f_g}\right)$$

$$M_1 \text{ is} > -\frac{Ar^2 f_{cb}}{y_b}$$

$$\text{i.e., } y_{1i} \text{ is} > y_{1f} \text{ if } M_1 > -f_{cb}Z_b$$

Similar relationships can be established for $y_1$ when the controlling limiting stress is the limiting tensile (or minimum compression) stress at face a, and for $y_2$. They may be utilized as summarized in Table 1 to determine which value of prestress force should be used when calculating $y_1$ and $y_2$.

The rules in Table 1 are general. In the case of a continuous T-beam, the controlling stresses will usually be at the bottom face of the section; in this case, the rules reduce to:

For $y_1$, use $P_i$ if $M_1$ is $> -f_{cb}Z_b$, otherwise use $P$.

For $y_2$, use $P_i$ if $M_2$ is $< -f_{tb}Z_b$, otherwise use $P$.

## EXAMPLE B

The T-beam shown in Fig. B1 is part of a floor system continuous over three spans AB, BC and CD of 60 ft (18.3 m) each. It carries a live load of 100 psf (4.79 kPa), which may act on all spans, alternate spans or any two adjacent spans. The dead weight of the beam is 435 lbs per ft (647 kg/m) of span. The allowable stresses are 2250 psi (15.5 MPa) compression and 425 psi (2.93 MPa) tension, both initially and finally. Based on consideration of the range of moments to be resisted by the section and compensation for the effects of permanent loads, the initial and final prestress forces are $P_i$ = 345 kips (1535 kN) and $P$ = 299 kips (1330 kN). Determine a suitable location for the c.g.s. along the length of the beam.

The required section modulus was determined using the approximate equation proposed by Magnel:[2]

$$\text{Required } Z = \frac{M}{0.775 f_c + f_t}$$

where $M$ is the maximum range of moment, $f_c$ is the allowable compressive stress and $f_t$ is the allowable tensile stress at the flexural tensile face under load.

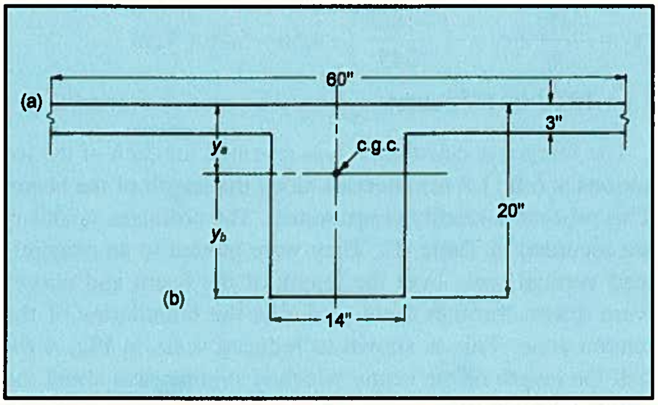


Fig. B1. Beam cross section.

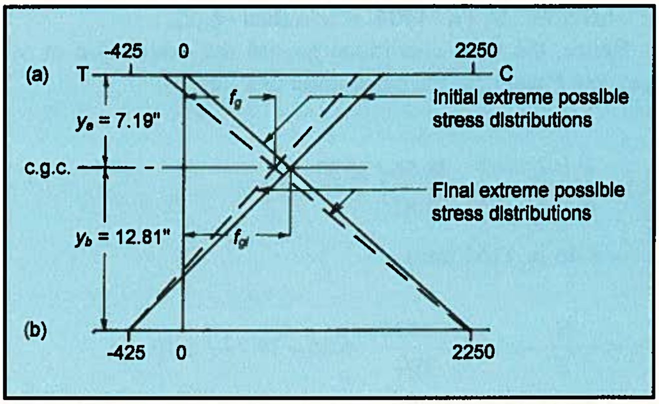


Fig. B2. Initial and final extreme possible stress distributions.

Table B1 summarizes the dead load, live load and total moments at 6 ft (1.8 m) intervals along the beam, measured from the center of the middle span BC. (Also shown are the later entered, calculated ordinates $y_1$ and $y_2$ of the boundaries of the tendon zone, within which the line of pressure must lie if the allowable stresses are not to be exceeded under any loading condition.)

### Section Properties (Fig. B1)

$y_a = 7.19$ in. (183 mm)
$y_b = 12.81$ in. (325 mm)
$A = 418$ sq in. (0.27 m²)
$Z_a = 2241$ cu in. (0.0367 m³)
$Z_b = 1258$ cu in. (0.021 m³)
$r^2 = I/A = 38.56$ sq in. (0.025 m²)

### Typical Calculation of $y_1$ and $y_2$

Using:
$P_i = 345$ kips (1535 kN)
$P = 299$ kips (1330 kN)
$f_c = 2250$ psi (15.5 MPa)
$f_t = -425$ psi (–2.93 MPa )
$f_{gi} = P_i/A = 345000/418 = 825$ psi (5.69 MPa)
$f_g = P/A = 299000/418 = 715$ psi (4.93 MPa)
$Z_a = 2241$ cu in. (0.0367 m³)
$Z_b = 1258$ cu in. (0.021 m³)
At B and C:
$M_1 = (-367)(12) = -4404$ kip-in. (498 kN-m)
$M_2 = (-157)(12) = -1884$ kip-in. (213 kN-m)
$y_1$: Bottom limiting compressive stress can be reached for both $P = P_i$ and $P = P$.
$-f_{cb}Z_b = -(2.250)(1258) = -2831$ kip-in. (–320 kN-m)

Table B1. Summary of moments (kip-ft) and values for $y_1$ and $y_2$ (in.).

| Location | x (ft) | Dead load moment | Maximum positive LL moment | Maximum negative LL moment | Maximum positive or minimum negative total moment, $M_2$ | Maximum negative or minimum positive total moment, $M_1$ | $y_1$ | $y_2$ |
|---|---|---|---|---|---|---|---|---|
| Midspan BC | 0 | +39 | +135 | -90 | +174 | -51 | -3.43 | -2.18 |
| | 6 | +31 | +126 | -90 | +157 | -59 | -3.14 | -1.50 |
| | 12 | +8 | +99 | -90 | +107 | -81 | -2.38 | +0.50 |
| | 18 | -32 | +54 | -90 | +22 | -122 | -0.95 | +3.79 |
| | 24 | -86 | 0 | -114 | -86 | -200 | +1.76 | +7.55 |
| B or C | 30 | -157 | 0 | -210 | -157 | -367 | +8.27 | +10.02 |
| | 36 | -70 | 0 | -108 | -70 | -178 | +1.00 | +6.99 |
| | 42 | -1 | +72 | -72 | +71 | -73 | -2.66 | +1.95 |
| | 48 | +55 | +126 | -63 | +181 | -8 | -4.92 | -2.47 |
| | 54 | +94 | +162 | -54 | +256 | +40 | -6.59 | -5.48 |
| | 60 | +118 | +180 | -45 | +298 | +73 | -7.74 | -7.16 |
| | 66 | +125 | +180 | -36 | +305 | +89 | -8.29 | -7.44 |
| | 72 | +118 | +162 | -27 | +280 | +91 | -8.36 | -6.44 |
| | 78 | +94 | +126 | -18 | +220 | +76 | -7.84 | -4.03 |
| | 84 | +55 | +72 | -9 | +127 | +46 | -6.80 | -0.30 |
| A or D | 90 | 0 | 0 | 0 | 0 | 0 | -5.20 | +4.56 |

**Note:** 1 in. = 25.4 mm; 1 kip-ft = 1.356 kN-m.

Therefore, $M_1$ (= –4404) is less than $-f_{cb}Z_b$.

Hence, the final conditions govern the calculation of $y_1$, i.e., use $P$ and $f_g$ in the calculation of $c'$ and $y_1$:

$$c' = \frac{r^2}{y_b}\left(\frac{f_b}{f_g} - 1\right) = \frac{38.56}{12.81}\left(\frac{2250}{715} - 1\right)$$

= 6.46 in. (164 mm)

$$y_1 = -\frac{M_1}{P} - c' = -\frac{-4404}{299} - 6.46 = 14.73 - 6.46$$

= +8.27 in. (210 mm)

$y_2$: Bottom limiting tensile stress can be reached for both $P = P_i$ and $P = P$.

$-f_{tb}Z_b$ = –(–0.425)(1258) = +535 kip-in. (60.5 kN-m)

Therefore, $M_2$ (= –1884) is less than $-f_{tb}Z_b$.

Hence, the initial conditions govern the calculation of $y_2$, i.e., use $P_i$ and $f_{gi}$ in the calculation of $c$ and $y_2$:

$$c_i = \frac{r^2}{y_b}\left(1 - \frac{f_b}{f_{gi}}\right) = \frac{38.56}{12.81}\left(1 - \frac{-425}{825}\right)$$

= 4.56 in. (116 mm)

$$y_2 = -\frac{M_2}{P_i} + c_i = -\left(\frac{-1884}{345}\right) + 4.56 = 5.46 + 4.56$$

= 10.02 in. (255 mm)

The foregoing calculation was repeated for each of the locations at 6 ft (1.8 m) intervals along the length of the beam. This process is readily programmed. The ordinates $y_1$ and $y_2$ are recorded in Table B1. They were plotted to an exaggerated vertical scale over the length of the beam and curves were drawn through them to define the boundaries of the tendon zone. This is shown to reduced scale in Fig. 4 for half the length of the beam, which is symmetrical about the middle of Span BC.

### Design of the Line of Pressure (the c.g.s. line)

A suitable line of pressure can be developed by trial and error, adjusting the shape of the line until the displacements $u_p$ at the moment releases at Supports B and C are zero. The secondary moments due to prestress will then be zero. The final c.g.s. line is arrived at by transforming the line of pressure so that the requisite amount of cover is provided for the tendons at critical sections.

In this case, the displacements $u_p$ at the moment releases at B and C will be equal, due to symmetry, if the line of pres-

Table B2. Design of line of pressure.

| Location | $x$ (ft) | $M_1$ | $q$ | $qM$ | $(M_0)_p/P = e_s$ (in.) first trial | $\frac{qM_1(M_0)_p}{P}$ (kip-in.²) | Adjusted $e_s$ (in.) | $\frac{qM_1(M_0)_p}{P}$ adjusted (kip-in.²) | $\delta$ * |
|---|---|---|---|---|---|---|---|---|---|
| C | –30 | 0.00 | 1 | 0.00 | 8.27 | 0.00 | 8.27 | 0.00 | 0.00 |
| | –24 | –0.10 | 4 | –0.40 | 4.70 | –1.88 | 5.50 | –2.20 | –0.32 |
| | –18 | –0.20 | 2 | –0.40 | 1.42 | –0.57 | 2.00 | –0.80 | –0.23 |
| | –12 | –0.30 | 4 | –1.20 | –0.94 | 1.13 | –0.94 | 1.13 | 0.00 |
| | –6 | –0.40 | 2 | –0.80 | –2.55 | 2.04 | –2.55 | 2.04 | 0.00 |
| Midspan | 0 | –0.50 | 4 | –2.00 | –3.43 | 6.86 | –3.43 | 6.86 | 0.00 |
| | 6 | –0.60 | 2 | –1.20 | –2.55 | 3.06 | –2.55 | 3.06 | 0.00 |
| | 12 | –0.70 | 4 | –2.80 | –0.94 | 2.63 | –0.94 | 2.63 | 0.00 |
| | 18 | –0.80 | 2 | –1.60 | 1.42 | –2.27 | 2.00 | –3.20 | –0.93 |
| | 24 | –0.90 | 4 | –3.60 | 4.70 | –16.92 | 5.50 | –19.80 | –2.88 |
| B | 30 | –1.00 | 2 | –2.00 | 8.27 | –16.54 | 8.27 | –16.54 | 0.00 |
| | 36 | –0.90 | 4 | –3.60 | 4.50 | –16.20 | 5.01 | –18.04 | –1.84 |
| | 42 | –0.80 | 2 | –1.60 | –0.36 | 0.58 | 0.91 | –1.46 | –2.03 |
| | 48 | –0.70 | 4 | –2.80 | –3.30 | 9.24 | –2.58 | 7.22 | –2.02 |
| | 54 | –0.60 | 2 | –1.20 | –5.50 | 6.60 | –5.49 | 6.59 | –0.01 |
| Midspan | 60 | –0.50 | 4 | –2.00 | –7.16 | 14.32 | –7.16 | 14.32 | 0.00 |
| | 66 | –0.40 | 2 | –0.80 | –7.50 | 6.00 | –7.50 | 6.00 | 0.00 |
| | 72 | –0.30 | 4 | –1.20 | –7.05 | 8.46 | –7.05 | 8.46 | 0.00 |
| | 78 | –0.20 | 2 | –0.40 | –5.94 | 2.38 | –5.94 | 2.38 | 0.00 |
| | 84 | –0.10 | 4 | –0.40 | –3.39 | 1.36 | –3.39 | 1.36 | 0.00 |
| A | 90 | 0.00 | 1 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| | | | | | Summations = | 10.27 | | 0.01 | –10.26 |

**Note:** 1 in. = 25.4 mm; 1 ft = 0.3048 m; 1 kip-in.² = 2870 kN-mm².

* $\delta$ equals change in $[qM_1(M_0)_p]/P$ due to change in ordinate of line of pressure, $e_s$.

sure is made symmetrical about the center of Span BC. It is, therefore, only necessary to calculate, say, $(u_p)_1$. The integral may conveniently be evaluated using Simpson's Rule. For this purpose, the spans must be divided into an even number of parts of length $h$. In this example, each span is divided into 10 parts of width $h$ = 6 ft (1.8 m). We may write:

$$(u_p)_1 = \int_A^C \frac{M_1(M_0)_p}{EI} ds = \frac{h}{3} \sum_A^C \frac{qM_i(M_0)_p}{EI}$$

$$= \frac{h}{3} \frac{2P}{EI} \sum_A^C \frac{qM_1(M_0)_p}{P}$$

where $q$ = Simpson's Rule coefficient (1,4,2,4,2,4,—— 2,4,1).

To make $(u_p)_0$ equal to zero, it is, therefore, only necessary to make:

$$\sum_A^C qM_1 \frac{(M_0)_p}{P}$$

equal to zero, where $(M_0)_p/P = e_s$, the ordinate of the line of pressure.

The process is readily carried out in tabular form and may be expedited using a spreadsheet. This is shown in Table B2.

Because:

$$\sum_A^C \frac{qM_1(M_0)_p}{P} \approx 0$$

then $(u_p)_1 \approx 0$ and by symmetry $(u_p)_2 \approx 0$.

Therefore, the adjusted line is a line of pressure. If the c.g.s. line coincided with this line, there would be no secondary moments due to prestress. However, the actual c.g.s. line must be a downward transformation of this line in order to provide adequate cover for the tendon. The line of pressure would remain unchanged and, therefore, the allowable concrete stresses would not be exceeded along the entire length of the beam at any time.

To provide adequate cover, the c.g.s. line will be transformed downward by 4.39 in. (112 mm) at B and C, and proportionately at other locations. The c.g.s. line is shown in Fig. 4. It has eccentricities of –9.36 in. (–238 mm) at Midspan AB, –7.82 in. (–199 mm) at Midspan BC and +3.88 in. (99 mm) at Supports B and C.

The downward transformation will result in secondary moments as follows:

(a) Throughout Span BC, $M_s = P_v = P(4.39)$ = (299) (4.39)/12 = 109.4 kip-ft (148 kN-m) after losses.

(b) In the outer Spans AB and CD, $M_s$ will vary linearly from zero at the outer supports to 109.4 kip-ft (148 kN-m) at the interior supports.

These secondary moments must be taken into account at ultimate, as discussed in the answer to Question 4 and as in Example C.

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**Q4:** *How do secondary moments due to prestress affect the behavior of a continuous prestressed concrete beam at ultimate?*

**A4:** As discussed in the answer to Q1, secondary moments due to prestress occur in a continuous beam because the beam is not free to deform due to its statical indeterminacy. If the beam were made statically determinate by inserting hinges at each interior support, the ends of the beams would rotate relative to one another when the beam was prestressed. (An example of this is shown in Fig. 8a, in which the c.g.s. line for the two-span beam has been transformed downward from the line of pressure.) The suppression of these relative end rotations causes the secondary moments.

If full redistribution of moments occurs at ultimate, i.e., if the full flexural capacity of all critical sections is developed, then just before failure the beam becomes statically determinate due to the inelastic rotations occurring at $n$ critical sections, where $n$ is the degree of indeterminacy. The end rotations due to prestressing are no longer suppressed and the secondary moments as such must disappear. However, the end rotations due to prestress must be combined with the end rotations due to gravity loads to obtain the required inelastic rotations in the hinging regions at interior supports. In the case of the two-span continuous beam of Figs. 8a and 8b, the required inelastic rotation at the center support will be reduced from $\gamma$ to $(\gamma - \theta)$.

In general, if the tendon profile has been transformed downward at an interior support, then the relative rotation of the ends of the beam segments meeting at that support caused by the prestressing moments is of opposite sign to the inelastic relative rotation, which must occur if the design support moment is less than the support moment due to gravity loads when behavior is completely elastic. The net amount of inelastic rotation necessary in a hinging region in order that a particular amount of moment redistribution can occur is, therefore, reduced by the relative end rotation caused by transforming the tendon profile downward.

Conversely, if a given amount of inelastic rotational capacity is available in a support hinging region, then the

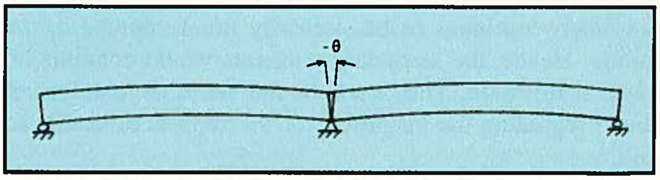


Fig. 8a. Relative end rotation due to prestress when continuous beam is made determinate by inserting a hinge at the interior support.

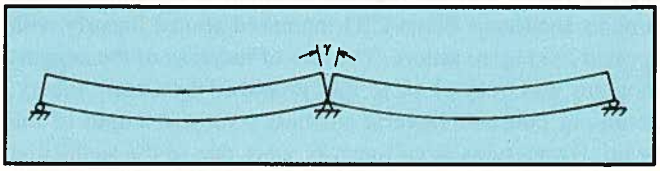


Fig. 8b. Relative end rotation due to gravity loads at ultimate just before failure of the midspan sections.

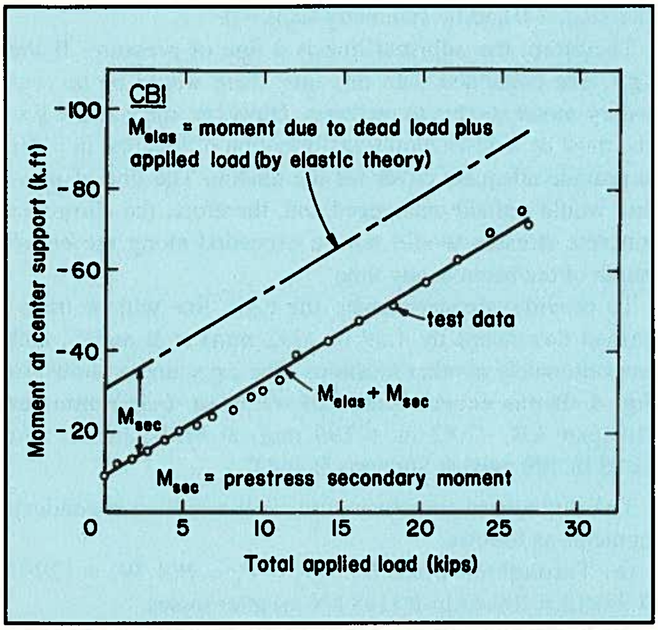


Fig. 9. Variation of center support moment with applied load in test of the two-span continuous prestressed concrete T-beam CB1.

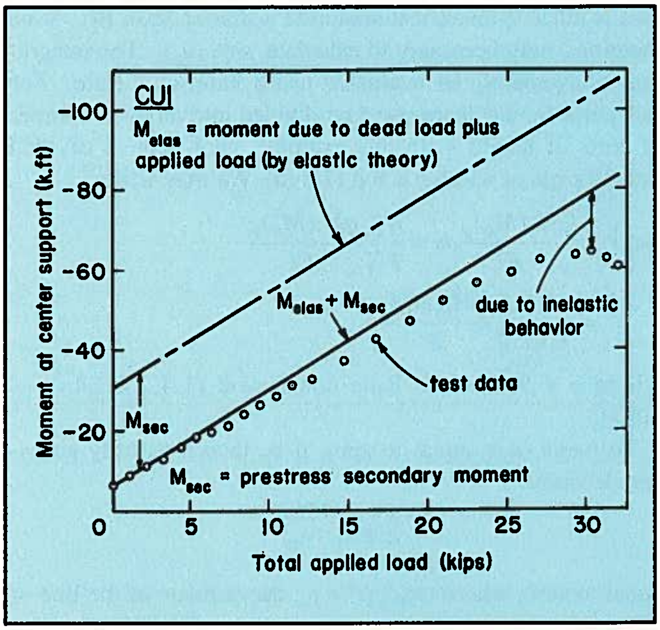


Fig. 10. Variation of center support moment with applied load in test of the two-span continuous prestressed concrete T-beam CU1.

amount of moment redistribution possible is increased. The amount of the increase is equal to the positive secondary moment due to prestress caused in the elastic continuous beam. This is because the secondary moment, and the amount by which the design support moment can be varied from the elastic theory moment due to gravity loads, are both related in the same way to the relative end rotations of the beam segments meeting at the support.

If the flexural strengths of the critical sections of a prestressed continuous beam are such that no inelastic rotation of the hinging regions occurs before ultimate, the foregoing discussion leads to the conclusion that, in this case, the support moments at failure will be the elastic theory negative moment due to factored gravity loads plus the positive secondary moment due to prestressing (assuming a downwardly transformed tendon profile). An alternative view of this special case would be that because no inelastic rotation of hinging regions occurred before ultimate, the beam continues to be statically indeterminate up to failure. Hence, the secondary moments would continue to exist at ultimate. This leads to the same conclusion as above regarding the magnitude of the support moments at failure.

Behavior very similar to this occurred in the test[3] of a prestressed concrete T-beam (CB1), which was continuous over two spans of 28 ft (8.53 m) each. Both loads and reactions were monitored continuously during the test. It can be seen in Fig. 9 that the bending moment at the intermediate support section of Beam CB1 increased almost linearly with applied load up to failure. The rate of increase of the support moment was very close to that predicted by elastic theory, assuming constant flexural stiffness over the length of the beam. (Departures from linearity were due to the sequential occurrence of flexural cracking at various locations along the beam.)

It can be seen from the behavior of this beam that the secondary moment due to prestressing does not change, even though the force in the prestressed reinforcement increases when the applied load is increased. This is because the secondary moments result from the suppression of deformations due to prestressing of the tendons. The subsequent increase in tendon force is due to deformation of the beam due to applied loads. The increased tendon force does not produce any additional deformation of the beam and, hence, there is no increase in secondary moment due to suppression of additional deformation due to prestress.

In the general case, for a beam in which the tendon profile has been transformed downwards, the negative support moment at failure is the elastic theory support moment due to factored dead and live loads, minus the positive secondary moment due to prestressing, minus the redistribution of moments possible as a result of inelastic behavior of the beam in the region of the support. This type of behavior was observed[3] in the case of the continuous Beam CU1, as shown in Fig. 10.

The behavior described above is the reason for the wording of Section 18.10.3 and part of the last paragraph of R18.10.3 of the ACI Building Code (ACI 318-95) and Commentary (ACI 318R-95).[4]

> **18.10.3** — Moments to be used to compute required strength shall be the sum of the moments due to reactions induced by prestressing (with a load factor of 1.0) and the moments due to factored loads. Adjustment of the sum of these moments shall be permitted as allowed in 18.10.4.

> **R18.10.3** — To determine the moments used in design, the order of calculation should be: (a) determine moments due to dead and live load; (b) modify by algebraic addition of secondary moments; (c) redistribute as permitted.

### EXAMPLE C

For the beam of Example B, calculate the required strength at the support sections at B and C.

The service load moments at this section are: due to dead load, $D = -157$ kip-ft (–213 kN-m), due to live load, $L = -210$ kip-ft (–285 kN-m). The required flexural strength $M_u$ according to Section 18.10.3, is therefore given by:

$$M_u = 1.4D + 1.7L + M_s$$

The secondary moment due to prestress $M_s = +109.4$ kip-ft (148 kN-m).

Hence:

$$\begin{aligned} M_u &= 1.4(-157) + 1.7(-210) + (+109.4) \\ &= -219.8 - 357.0 + 109.4 \\ &= -467.4 \text{ kip-ft } (634 \text{ kN-m}) \end{aligned}$$

This moment could be further reduced if necessary by making use of the moment redistribution provisions of Section 18.10.4.

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4. ACI Committee 318, "Building Code Requirements for Structural Concrete (ACI 318-95) and Commentary (ACI 318R-95)," American Concrete Institute, Farmington Hills, MI, 1995.



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### NOTATION

$A$ = area of cross section
$c$ = distance from c.g.c. to upper boundary of limiting zone
$c'$ = distance from c.g.c. to lower boundary of limiting zone
c.g.c. = centroid of cross section
c.g.s. = center of action of prestressing force
$e$ = eccentricity of prestress force with respect to centroid of section
$F$ = concentrated force acting on concrete within a span, due to angular discontinuity in tendon
$f_a$ = stress at top edge of section
$f_b$ = stress at bottom edge of section
$f_{ca}$ = allowable compressive stress at top edge of section
$f_{cb}$ = allowable compressive stress at bottom edge of section
$f_g$ = compressive stress at centroid of section due to prestress after all losses
$f_{gi}$ = compressive stress at centroid of section due to initial prestress
$f_{ta}$ = allowable tensile or minimum compressive stress at top edge of section
$f_{tb}$ = allowable tensile or minimum compressive stress at bottom edge of section
$h$ = width of element in Simpson's Rule calculation
$I$ = moment of inertia of cross section about its centroid
$L$ = span length
$M$ = moment
$(M_0)_p$ = prestressing moment $Pe$ in beam made statically determinate
$M_p$ = prestressing moment = $Pe + M_s$
$M_s$ = secondary moment due to prestressing
$P$ = prestressing force after losses
$P_i$ = initial prestressing force
$Q$ = concentrated force acting on concrete over support, due to angular discontinuity in tendon
$q$ = intensity of radial force on concrete from tendon, equal to $P/r$
$q$ = Simpson's Rule coefficient
$R$ = restraint force at support due to action of prestress force
$r$ = radius of gyration of cross section
$r$ = radius of curvature of tendon
$V$ = vertical component of tendon force at each end of beam
$v$ = displacement from c.g.s. of effective point of action of prestressing force due to restraint of deformation of beam due to prestressing
$y_1$ = distance from c.g.c. to lower boundary of tendon zone
$y_2$ = distance from c.g.c. to upper boundary of tendon zone
$y_a$ = distance from c.g.c. to top edge of cross section
$y_b$ = distance from c.g.c. to bottom edge of cross section
$Z_a$ = section modulus with respect to top edge of section
$Z_b$ = section modulus with respect to bottom edge of section